EM 3 Section 14: Electromagnetic Energy and the Poynting Vector

*14. 1. Poynting's Theorem (*Griffiths 8.1.2*)*

Recall that we saw that the total energy stored in electromagnetic fields is:

$$U = U_M + U_E = \frac{1}{2}\int_{allspace}\left(\frac{1}{\mu_0}B^2 + \epsilon_0 E^2\right)\mathrm{d}V \tag{1}$$

Let us now derive this more generally. Consider some distribution of charges and currents. In small time $\mathrm{d}t$ a charge will move $\underline{v}\mathrm{d}t$ and, according to the Lorentz force law, the work done on the charge will be

$$\mathrm{d}U = \underline{F}\cdot\underline{\mathrm{d}l} = q(\underline{E} + \underline{v}\times\underline{B})\cdot\underline{v}\mathrm{d}t = q\underline{E}\cdot\underline{v}\mathrm{d}t$$

where as usual the magnetic forces do no work. Now let $q = \rho\mathrm{d}V$ (usual definition of charge density) and $\rho\underline{v} = \underline{J}$ (usual definition of current). Then dividing through by $\mathrm{d}t$ and integrating over a volume V containing the charges, we find that the rate at which work is done (i.e. the power delivered to the system) is

$$\boxed{\frac{\mathrm{d}U}{\mathrm{d}t} = \int_V \underline{E}\cdot\underline{J}\,\mathrm{d}V} \tag{2}$$

Thus $\underline{E}\cdot\underline{J}$ is the power delivered per unit volume. Now use MIV to express

$$\underline{E}\cdot\underline{J} = \frac{1}{\mu_0}\underline{E}\cdot(\underline{\nabla}\times\underline{B}) - \epsilon_0\underline{E}\cdot\frac{\partial\underline{E}}{\partial t}$$

Furthermore we can use a product rule from lecture 1 to write

$$\begin{aligned}\underline{E}\cdot(\underline{\nabla}\times\underline{B}) &= \underline{B}\cdot(\underline{\nabla}\times\underline{E}) - \underline{\nabla}\cdot(\underline{E}\times\underline{B})\\ &= -\underline{B}\cdot\frac{\partial\underline{B}}{\partial t} - \underline{\nabla}\cdot(\underline{E}\times\underline{B})\end{aligned}$$

where we used MIII in the last line. Putting it all together, and noting

$$\underline{B}\cdot\frac{\partial\underline{B}}{\partial t} = \frac{1}{2}\frac{\partial B^2}{\partial t} \qquad \underline{E}\cdot\frac{\partial\underline{E}}{\partial t} = \frac{1}{2}\frac{\partial E^2}{\partial t}\ ,$$

yields

$$\underline{E}\cdot\underline{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\underline{\nabla}\cdot(\underline{E}\times\underline{B})$$

Finally we can integrate over the volume V containing the currents and charges and use the divergence theorem on the second term to obtain from (2)

$$\boxed{\frac{\mathrm{d}U}{\mathrm{d}t} = -\frac{\partial}{\partial t}\int_V \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\mathrm{d}V - \frac{1}{\mu_0}\oint_S(\underline{E}\times\underline{B})\cdot\underline{\mathrm{d}S}} \tag{3}$$

Let us now examine each term in **Poynting's Theorem** (3): the left hand side is the power delivered to the volume i.e. the rate of *gain* in energy of the particles; the first term on the right hand side is the rate of *loss* of electromagnetic energy stored in fields *within* the volume; the second term is the rate of energy transport *out* of the volume i.e. across the surface S.

Thus Poynting's theorem reads: energy *lost* by fields = energy *gained* by particles+ energy flow out of volume. Hence we can identify the vector

$$\boxed{\underline{S} = \frac{1}{\mu_0}\underline{E} \times \underline{B}} \tag{4}$$

as the **energy flux density** (energy per unit area per unit time) and it is known as the **Poynting vector** (it 'Poynts' in the direction of energy transport).

Also we can write Poynting's theorem as a continuity equation for the total energy $U = U_{em} + U_{mec}$. The left hand side of (3) is the rate of change of *mechanical energy* thus

$$\frac{\mathrm{d}(U_{em} + U_{mec})}{\mathrm{d}t} = -\oint_S \underline{S} \cdot \mathrm{d}\underline{A}$$

(to avoid a nasty clash of notation with $\underline{S}$ as Poynting vector we use $\mathrm{d}\underline{A}$ rather than $\mathrm{d}\underline{S}$ as vector element of area). As usual, expressing energy as a volume over energy *densities* u_{em},u_{mec} and using the divergence theorem on the right hand side we arrive at

$$\boxed{\frac{\partial}{\partial t}(u_{em} + u_{mec}) = -\underline{\nabla} \cdot \underline{S}} \tag{5}$$

which is the continuity equation for energy density. Thus the Poynting vector represents the flow of energy in the same way that the current $\underline{J}$ represents the flow of charge.

*14. 2. Energy of Electromagnetic Waves (*Griffiths 9.2.3*)*

As we saw last lecture a monochromatic plane wave in vacuo propagating in the $\underline{e}_z$ direction is described by the fields:

$$\underline{E} = \underline{e}_x E_0 \cos(kz - \omega t) \qquad \underline{B} = \underline{e}_y B_0 \cos(kz - \omega t) \tag{6}$$

where

$$B_0 = \frac{E_0}{c}$$

The total energy stored in the fields associated with the wave is:

$$U = U_E + U_M \quad = \quad \frac{1}{2}\int_V \left(\frac{B^2}{\mu_0} + \epsilon_0 E^2\right) \mathrm{d}V$$

Now since $|\underline{B}| = |\underline{E}|/c$ and $c = 1/\sqrt{\mu_0\epsilon_0}$ we see that the electric and magnetic contributions to the total energy are equal and the electromagnetic energy *density* is (for a linearly polarised wave)

$$u_{EM} = \epsilon_0 E^2 = \epsilon_0 E_0^2 \cos^2(kz - \omega t)$$

The Poynting vector becomes for monochromatic waves

$$\underline{S} = \frac{1}{\mu_0}(\underline{E} \times \underline{B}) = c\epsilon_0 E_0^2 \cos^2(kz - \omega t)\underline{e}_z = u_{EM} c \underline{e}_z$$

Note that $\underline{S}$ is just the energy density multiplied by the velocity of the wave $c\underline{e}_z$ as it should be. Generally

$$\underline{S} = u_{EM} c \hat{\underline{k}}$$

N.B To compute the Poynting vector it is simplest to use a real form for the fields $\underline{B}$ and $\underline{E}$ rather than a complex exponential representation.

The time average of the energy density is is defined as the average over one period T of the wave

$$\begin{aligned}\langle u_{EM}\rangle &= \frac{\epsilon_0 E_0^2}{T}\int_0^T \cos^2(kz-\omega t)\mathrm{d}t \\ &= \frac{\epsilon_0 E_0^2}{T}\frac{T}{2} = \frac{1}{2}\epsilon_0 E_0^2 = \frac{1}{2}\frac{B_0^2}{\mu_0}\end{aligned}$$

The energy density of an electromagnetic wave is proportional to the square of the amplitude of the electric (or magnetic) field.

14. 3. Example of discharging capacitor

Consider a discharging circular parallel plate capacitor (plates area A) in a circuit with a

Figure 1: Discharging capacitor in a circuit with a resistor

resistor R. Ohm's law gives

$$V_d = \frac{Q}{C} = IR$$

or

$$I = -\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{Q}{RC} \quad \Rightarrow \quad Q = Q_0 \mathrm{e}^{-t/RC} \quad I = \frac{Q_0}{RC}\mathrm{e}^{-t/RC}$$

Now assume a 'quasistatic' approximation where we treat the fields as though they were static:

$$\underline{E} = -\frac{Q}{A\epsilon_0}\hat{\underline{n}} = -\frac{Q}{A\epsilon_0}\mathrm{e}^{-t/RC}$$

We take the normal to the plates (direction of $\underline{E}$) is $\hat{\underline{n}}$. Now we can compute $\underline{B}$ through Ampère-Maxwell noting that the cylindrical symmetry implies that $\underline{B}$ is circumferential. The Amperian loop is a circle radius r between the capacitor plates where $\underline{J} = 0$

$$\oint \underline{B}\cdot \mathrm{d}\underline{l} = \mu_0 \int_S \left(\underline{J} + \epsilon_0 \frac{\partial \underline{E}}{\partial t}\right)\cdot \mathrm{d}\underline{S} = -\mu_0 \pi r^2 \epsilon_0 \frac{\partial}{\partial t}\left(\frac{Q_0}{A\epsilon_0}\mathrm{e}^{-t/RC}\right)$$

The lhs $= 2\pi r B_\phi$ so

$$\underline{B} = \frac{\mu_0 I(t) r}{2A}\underline{e}_\phi$$

The Poynting vector is given by

$$\underline{S} = \frac{1}{\mu_0}\underline{E} \times \underline{B} = -\frac{Q}{A\epsilon_0}\mathrm{e}^{-t/RC}\,\mu_0 I_0 \frac{r}{2A}\mathrm{e}^{-t/RC}\underline{e}_z \times \underline{e}_\phi = \frac{I_0^2 CR}{2A^2\epsilon_0}r\mathrm{e}^{-2t/RC}\underline{e}_r$$

Thus the Poynting vector and the direction of energy flow point *radially out of the capacitor.*

*14. 4. *Momentum of electromagnetic radiation*

Let us reinterpret the Poynting vector from a quantum perspective. Due to wave-particle duality, radiation can be thought of as photons travelling with speed c with energy

$$\varepsilon = \hbar\omega = h\nu$$

The momentum of a single photon

$$\underline{p} = \hbar\underline{k} = \frac{\varepsilon}{c}\hat{\underline{k}}$$

For n photons per unit volume travelling at speed c we can interpret the average Poynting vector as average energy density $n\varepsilon$ multiplied by velocity vector $c\hat{\underline{k}}$

$$\langle\underline{S}\rangle = n\varepsilon c\hat{\underline{k}} = \langle u_{EM}\rangle c\hat{\underline{k}}$$

Again thinking of the energy transport as effected by photons, we must have an accompanying **momentum flux** $\underline{\tilde{P}}$

$\underline{\tilde{P}}$ is defined as the momentum carried across a plane normal to propagation, per unit area per unit time

For each photon $p = \varepsilon/c$ (along $\hat{\underline{k}}$) so

$$\underline{\tilde{P}} = \underline{S}/c$$

If light strikes the absorber (normal incidence) momentum is absorbed, this creates a force per unit area equal to the incoming (normal) momentum flux

This causes radiation pressure

$$p_{rad} = \underline{\tilde{P}} \cdot \hat{\underline{n}} = S/c \quad \Rightarrow \quad p_{rad} = \langle u_{EM}\rangle$$

If light is reflected not absorbed so twice the momentum is imparted, p_{rad} doubles but so does $\langle u_{EM}\rangle$, and this result still holds.

To understand radiation pressure classically let's go back to the example of an x polarised wave propagating in $\underline{e}_z$ direction: the electric field moves charges, on the surface the radiation strikes, in the x direction; then the Lorentz force $q\underline{v} \times \underline{B}$ (with $\underline{v}$ in the x direction and $\underline{B}$ in the y direction) is in the $\underline{e}_z$ direction and creates the pressure.

Above is for a **collimated** light beam (i.e. single direction) The other extreme is "diffuse radiation" = light bouncing around in all directions; this gives instead

$$p_{rad} = \langle u_{EM}\rangle/3$$

(the factor 1/3 is as in kinetic theory of gases).