New methods in lattice hadron spectroscopy

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Overview

- Motivation - open questions in spectroscopy
- What is needed for precision spectroscopy?
- Distillation: new approach to creating hadrons
- Spin on the lattice
- The problem with distillation...
- Spectroscopy calculations using distillation
- Conclusions and outlook

Gluonic excitations in light meson spectrum?

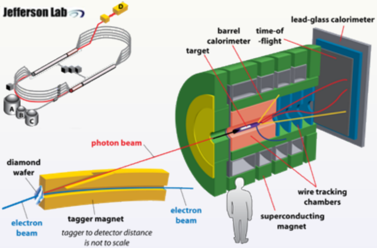
- QCD: constituent quarks and gluons are confined

constituents			quark model label
$3 \otimes \bar{3}$	$=$	$\mathbf{1} \oplus 8$	meson
$3 \otimes 3 \otimes 3$	$=$	$\mathbf{1} \oplus 8 \oplus 8 \oplus 10$	baryon
$8 \otimes 8$	$=$	$\mathbf{1} \oplus 8 \oplus 8 \oplus 10 \oplus \overline{10}$	glueball
$\bar{3} \otimes 8 \otimes 3$	$=$	$\mathbf{1} \oplus 8 \oplus 8 \oplus 8 \oplus 10 \oplus \overline{10}$	hybrid

Where are the gluonic excitations?

- A long-standing open experimental question - Compass, GlueX, PANDA, . . .
- Best option: "exotic" $J^{PC} = 0^{--}, \text{odd}^{-+}, \text{even}^{+-}$

JLab and GlueX



- 12 GeV upgrade to CEBAF ring
- New experimental hall: Hall D
- New experiment: GlueX

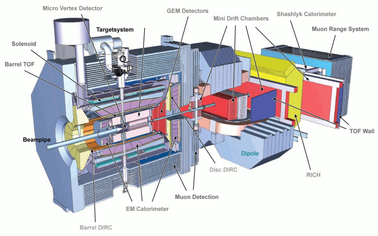
- Aim: photoproduce mesons, in particular the hybrid meson (with intrinsic gluonic excitations
- Expected to start taking data 2014
- Edinburgh involvement - ForwardTagger@JLab in CLAS 12

Panda@FAIR, GSI

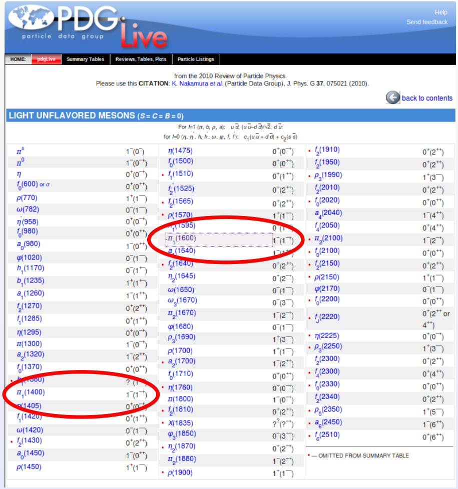
- Extensive new construction at GSI Darmstadt
- Expected to start operation 2014

PANDA: Anti-Proton ANnihilation at DArmstadt

- Anti-proton beam from FAIR on fixed-target.
- Physics goals include searches for hybrids and glueballs (as well as charm and baryon spectroscopy).



The PDG view



from the 2010 Review of Particle Physics.
Please use this **CITATION**: K. Nakamura *et al.* (Particle Data Group), J. Phys. G **37**, 075021 (2010).

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LIGHT UNFLAVORED MESONS ($S = C = B = 0$)

For I=1 (π, b, ρ, a): $u\bar{d}$, $(u\bar{u}-d\bar{d})/\sqrt{2}$, $d\bar{u}$;
for I=0 (η, η', h, h', ω, φ, f, f'): $c_1(u\bar{u}+d\bar{d}) + c_2(s\bar{s})$

$\pi^\pm$	$1^-(0^-)$	$\eta(1475)$	$0^+(0^{-+})$	• $f_2(1910)$	$0^+(2^{++})$
π^0	$1^-(0^{-+})$	$f_0(1500)$	$0^+(0^{++})$	$f_2(1950)$	$0^+(2^{++})$
η	$0^+(0^{-+})$	• $f_1(1510)$	$0^+(1^{++})$	• $\rho_3(1990)$	$1^+(3^{--})$
$f_0(600)$ or σ	$0^+(0^{++})$	$f_2'(1525)$	$0^+(2^{++})$	$f_2(2010)$	$0^+(2^{++})$
$\rho(770)$	$1^+(1^{--})$	• $f_2(1565)$	$0^+(2^{++})$	• $f_0(2020)$	$0^+(0^{++})$
$\omega(782)$	$0^-(1^{--})$	• $\rho(1570)$	$1^+(1^{--})$	$a_4(2040)$	$1^-(4^{++})$
$\eta'(958)$	$0^+(0^{-+})$	$\eta_1(1595)$	$0^+(1^{-+})$	$f_4(2050)$	$0^+(4^{++})$
$f_0(980)$	$0^+(0^{++})$	$\pi_1(1600)$	$1^-(1^{-+})$	• $\pi_2(2100)$	$1^-(2^{-+})$
$a_0(980)$	$1^-(0^{++})$	$a_1(1640)$	$1^-(1^{++})$	• $f_0(2100)$	$0^+(0^{++})$
$\varphi(1020)$	$0^-(1^{--})$	• $f_2(1640)$	$0^+(2^{++})$	• $f_2(2150)$	$0^+(2^{++})$
$h_1(1170)$	$0^-(1^{+-})$	$\eta_2(1645)$	$0^+(2^{-+})$	• $\rho(2150)$	$1^+(1^{--})$
$b_1(1235)$	$1^+(1^{+-})$	$\omega(1650)$	$0^-(1^{--})$	$\varphi(2170)$	$0^-(1^{--})$
$a_1(1260)$	$1^-(1^{++})$	$\omega_3(1670)$	$0^-(3^{--})$	• $f_0(2200)$	$0^+(0^{++})$
$f_2(1270)$	$0^+(2^{++})$	$\pi_2(1670)$	$1^-(2^{-+})$	• $f_J(2220)$	$0^+(2^{++}$ or $4^{++})$
$f_1(1285)$	$0^+(1^{++})$	$\varphi(1680)$	$0^-(1^{--})$	• $\eta(2225)$	$0^+(0^{-+})$
$\eta(1295)$	$0^+(0^{-+})$	$\rho_3(1690)$	$1^+(3^{--})$	• $\rho_3(2250)$	$1^+(3^{--})$
$\pi(1300)$	$1^-(0^{-+})$	$\rho(1700)$	$1^+(1^{--})$	$f_2(2300)$	$0^+(2^{++})$
$a_2(1320)$	$1^-(2^{++})$	• $a_2(1700)$	$1^-(2^{++})$	• $f_4(2300)$	$0^+(4^{++})$
$f_0(1370)$	$0^+(0^{++})$	$f_0(1710)$	$0^+(0^{++})$	• $f_0(2330)$	$0^+(0^{++})$
• $h_1(1380)$	$?^-(1^{+-})$	• $\eta(1760)$	$0^+(0^{-+})$	$f_2(2340)$	$0^+(2^{++})$
$\pi_1(1400)$	$1^-(1^{-+})$	$\pi(1800)$	$1^-(0^{-+})$	• $\rho_5(2350)$	$1^+(5^{--})$
$\eta(1405)$	$0^+(0^{-+})$	• $f_2(1810)$	$0^+(2^{++})$	• $a_6(2450)$	$1^-(6^{++})$
$f_1(1420)$	$0^+(1^{++})$	• $X(1835)$	$?^?(?^{-+})$	• $f_6(2510)$	$0^+(6^{++})$
$\omega(1420)$	$0^-(1^{--})$	$\varphi_3(1850)$	$0^-(3^{--})$		
• $f_2(1430)$	$0^+(2^{++})$	• $\eta_2(1870)$	$0^+(2^{-+})$	• — OMITTED FROM SUMMARY TABLE	
$a_0(1450)$	$1^-(0^{++})$	$\pi_2(1880)$	$1^-(2^{-+})$		
$\rho(1450)$	$1^+(1^{--})$	• $\rho(1900)$	$1^+(1^{--})$		

- PDG lists 77 light mesons
- 2 (!) 1^{-+} spin-exotics
- Most others fit into a quark model description
- Are there states with constituent gluons?
- Models: Different models disagree
- Lattice QCD can in principle provide answers directly from QCD lagrangian

Motivation

With evolving techniques, lattice QCD should shed light on questions such as:

- Last decade saw proliferation of new states above open-charm threshold. What are they?
- The quark model predicts many more baryon resonances than are seen. Why?
- Do hadronic molecules form? Tetraquarks?
- Are there intrinsic gluonic excitations in hadrons?
- Do glueballs exists as observable resonances?

Field theory on a Euclidean lattice


- Monte Carlo simulations are only practical using **importance sampling**
- Need a non-negative weight for each field configuration on the lattice

Minkowski $\longrightarrow$ Euclidean

- **Problem:** direct information on scattering is lost and must be inferred indirectly.
- **Benefit:** can isolate lightest states in the spectrum.
- For excitations and resonances, must use a **variational method.**

Quarks on the computer

- Most computer time spent handling quark dynamics
- Calculation of two-point correlator between isovector quark bilinears:

$$
\begin{aligned}
C(t) &= \frac{\int \mathcal{D}U\mathcal{D}\bar{\psi}\mathcal{D}\psi \; \bar{\psi}_u\Gamma^a\psi_d(t)\;\bar{\psi}_d\Gamma^b\psi_u(0)\;\; e^{-S_G[U]+\bar{\psi}_f M_f[U]\psi_f}}{\int \mathcal{D}U\mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{-S_G[U]+\bar{\psi}_f M_f[U]\psi_f}} \\
&= \frac{\int \mathcal{D}U\;\; \mathrm{Tr}\;\Gamma^a M_d^{-1}(t,0)\Gamma^b M_u^{-1}(0,t)\;\; \det M^2[U]\; e^{-S_G[U]}}{\int \mathcal{D}U\;\; \det M^2[U]\; e^{-S_G[U]}}
\end{aligned}
$$

- Quarks in lagrangian → **determinant**
- Quarks in measurement → **propagators**

Both present their own specific problems

Requirements for precision spectroscopy

- To study high-lying resonances requires:
  - Operators that create highly excited states
  - Operators that create multi-particle states
  - Precise data on energy shifts in finite L
  - Spin identification
- Need to exploit large **variational basis** of operators
- These requirements are hard to achieve with traditional lattice methods
- Need all elements of the quark propagator

Variational method in Euclidean QFT

- Ground-state energies found from $t \to \infty$ limit of:

Euclidean-time correlation function

$$\begin{aligned} c(t) &= \langle 0|\, \Phi(t) \quad \Phi^\dagger(0)\, |0\rangle \\ &= \sum_{k,k'} \langle 0|\, \Phi|k\rangle\langle k|e^{-\hat{H}t}|k'\rangle\langle k'|\Phi^\dagger\, |0\rangle \\ &= \sum_{k} |\langle 0|\, \Phi|k\rangle|^2 \quad e^{-E_k t} \end{aligned}$$

- So $\lim_{t\to\infty} c(t) = Ze^{-E_0 t}$
- Variational idea: find operator Φ to maximise $c(t)/c(t_0)$ from sum of basis operators $\Phi = \sum_a c_a \phi_a$

[C. Michael and I. Teasdale. NPB215 (1983) 433]
[M. Lüscher and U. Wolff. NPB339 (1990) 222]

Variational method

If we can measure $C_{ab}(t) = \langle 0|\phi_a(t)\phi_b^\dagger(0)|0\rangle$ for all a, b and solve generalised eigenvalue problem:

$$\mathbf{C}(t)\,\underline{v} = \lambda\,\mathbf{C}(t_0)\,\underline{v}$$

then

$$\lim_{t-t_0\to\infty} \lambda_k = e^{-E_k t}$$

For this to be practical, we need:

- a ‘good’ basis set that **resembles the states** of interest
- **all elements** of this correlation matrix measured

Efficient measurement

- Computing quark propagation in configuration generation and observable measurement is expensive.
- Objective: extract as much information from correlation functions as possible.

Two problems:

1. Most correlators: signal-to-noise falls exponentially
2. Making measurements can be costly:
    - Variational bases
    - Exotic states using more sophisticated creation operators
    - Isoscalar mesons
    - **Multi-hadron states**

- Good operators are **smeared**; helps with problem 1, can it help with problem 2?

- **Smeared field:** $\tilde{\psi}$ from ψ, the “raw” quark field in the path-integral:

$$\tilde{\psi}(t) = \square[U(t)]\ \ \psi(t)$$

- Extract the essential degrees-of-freedom.
- Smearing should preserve symmetries of quarks.
- Now form creation operator (e.g. a meson):

$$O_M(t) = \bar{\tilde{\psi}}(t)\Gamma\tilde{\psi}(t)$$

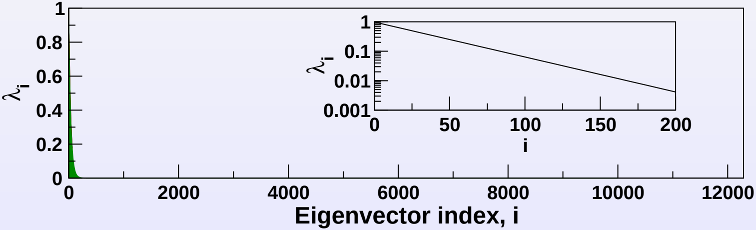
- Γ: operator in $\{\underline{s}, \sigma, c\} \equiv \{\text{position,spin,colour}\}$
- Smearing: overlap $\langle n|O_M|0\rangle$ is large for low-lying eigenstate $|n\rangle$

Gaussian smearing

- Many recipes in use. One popular gauge covariant choice is **gaussian** smearing:

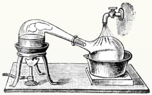
$$\lim_{n\to\infty}\left(1+\frac{\sigma\nabla^2}{n}\right)^n = \exp(\sigma\nabla^2)$$

- This acts in the space of coloured scalar fields on a time-slice: $N_s \times N_c$



- Data from $a_s \approx 0.12\text{fm}$ 16^3 lattice: $16^3 \times 3 = 12288$.

"*distill*: to extract the quintessence of" [OED]



- Distillation: **define** smearing to be explicitly a very low-rank operator. Rank is $N_{\mathcal{D}}(\ll N_s \times N_c)$.

Distillation operator

$$\Box(t) = V(t)V^{\dagger}(t)$$

with $V^a_{\underline{x},c}(t)$ a $N_{\mathcal{D}} \times (N_s \times N_c)$ matrix

- Example (used to date): $\Box_{\nabla}$ the **projection operator into $\mathcal{D}_{\nabla}$, the space spanned by the lowest eigenmodes of the 3-D laplacian**
- Projection operator, so idempotent: $\Box_{\nabla}^2 = \Box_{\nabla}$
- $\lim_{N_{\mathcal{D}} \to (N_s \times N_c)} \Box_{\nabla} = I$
- Eigenvectors of ∇^2 not the only choice...

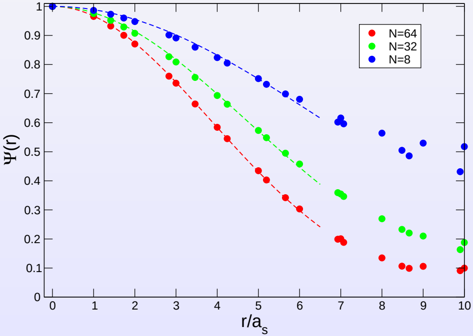
Distillation: preserve symmetries

- Using eigenmodes of the gauge-covariant laplacian **preserves lattice symmetries**

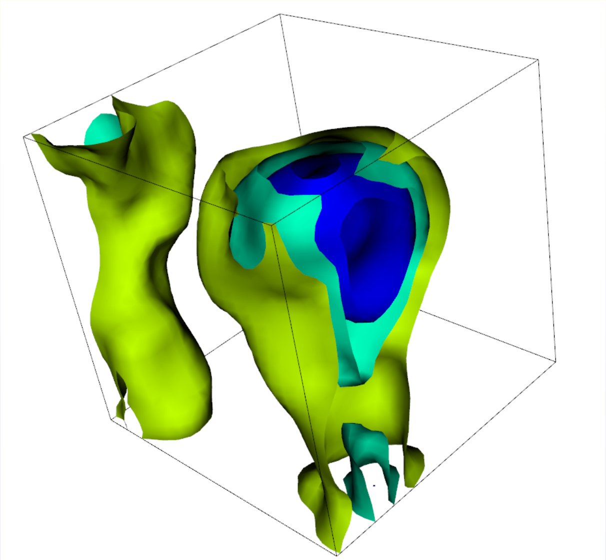
$$U_i(\underline{x}) \xrightarrow{g} U_i^g(\underline{x}) = g(\underline{x})U_i(\underline{x})g^\dagger(\underline{x}+\hat{\underline{\imath}})$$

$$\Box_\nabla(\underline{x},\underline{y}) \xrightarrow{g} \Box_\nabla^g(\underline{x},\underline{y}) = g(\underline{x})\Box_\nabla(\underline{x},\underline{y})g^\dagger(\underline{y})$$

- Translation, parity, charge-conjugation symmetric
- O_h symmetric
- Close to $SO(3)$ symmetric
- “local” operator



Eigenmodes of the laplacian



- Lowest mode on a $32^3 \equiv (3.8\ \mathrm{fm})^3$ lattice.

- Consider an isovector meson two-point function:

$$C_M(t_1 - t_0) = \langle\langle \bar{u}(t_1)\Box_{t_1}\Gamma_{t_1}\Box_{t_1} d(t_1) \quad \bar{d}(t_0)\Box_{t_0}\Gamma_{t_0}\Box_{t_0} u(t_0)\rangle\rangle$$

- Integrating over quark fields yields

$$C_M(t_1 - t_0) = \\ \langle \mathrm{Tr}_{\{\underline{s},\sigma,c\}} \left(\Box_{t_1}\Gamma_{t_1}\Box_{t_1} M^{-1}(t_1, t_0)\Box_{t_0}\Gamma_{t_0}\Box_{t_0} M^{-1}(t_0, t_1)\right)\rangle$$

- Substituting the low-rank distillation operator $\Box$ reduces this to a **much smaller** trace:

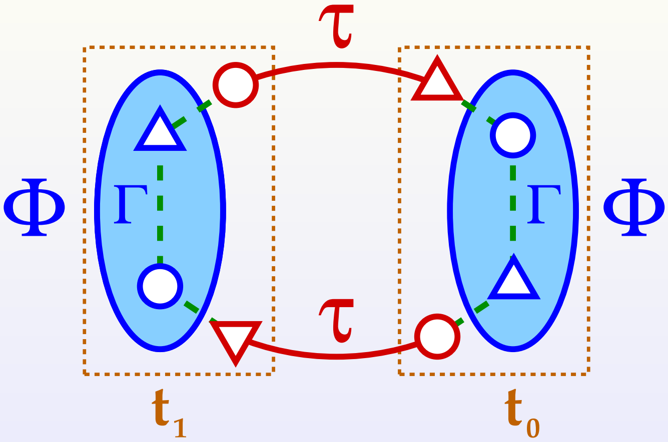
$$\boxed{C_M(t_1 - t_0) = \langle \mathrm{Tr}_{\{\sigma,\mathcal{D}\}} \left[\Phi(t_1)\tau(t_1, t_0)\Phi(t_0)\tau(t_0, t_1)\right]\rangle}$$

- $\Phi^{\alpha,a}_{\beta,b}$ and $\tau^{\alpha,a}_{\beta,b}$ are $(N_\sigma \times N_{\mathcal{D}}) \times (N_\sigma \times N_{\mathcal{D}})$ matrices.

$$\boxed{\Phi(t) = V^\dagger(t)\Gamma_t V(t)} \qquad \boxed{\tau(t, t') = V^\dagger(t) M^{-1}(t, t') V(t')}$$

The "perambulator"

Meson two-point function

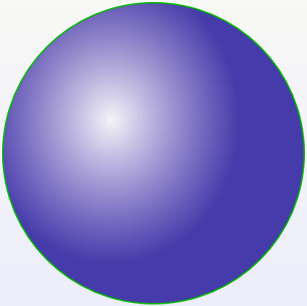


Distilled meson two-point correlation function

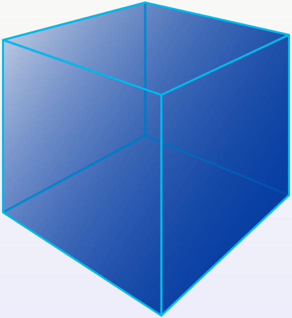
$$C_M(t_1 - t_0) = \mathrm{Tr}_{\{\sigma,D\}}\left[\Phi(t_1)\ \tau(t_1,t_0)\ \Phi(t_0)\ \tau(t_0,t_1)\right]$$

A tale of two symmetries

- Continuum: states classified by J^P irreducible representations of $O(3)$.



$O(3)$



O_h

- Lattice regulator breaks $O(3) \rightarrow O_h$
- Lattice: states classified by R^P **"quantum letter" labelling irrep of O_h**

Irreps of O_h

- O has 5 conjugacy classes (so O_h has 10)
- Number of conjugacy classes = number of irreps
- Schur: $24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2$
- These irreps are labelled A_1, A_2, E, T_1, T_2

	E	$8C_3$	$6C_2$	$6C_4$	$3C_2$
A_1	1	1	1	1	1
A_2	1	1	-1	-1	1
E	2	-1	0	0	2
T_1	3	0	-1	1	-1
T_2	3	0	1	-1	-1

Spin on the lattice

- O_h has 10 irreps: $\{A_1^{g,u}, A_2^{g,u}, E^{g,u}, T_1^{g,u}, T_2^{g,u}, \}$, where $\{g, u\}$ label even/odd parity.
- Link to continuum: subduce representations of $O(3)$ into O_h

	A_1	A_2	E	T_1	T_2
$J=0$	1				
$J=1$				1	
$J=2$			1		1
$J=3$		1		1	1
$J=4$	1		1	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

- Enough to search for degeneracy patterns in the spectrum? $4 \equiv 0 \oplus 1 \oplus 2$!

Example: $J^{PC} = 2^{++}$ meson creation operator

- Need more information to discriminate spins. Consider continuum operator that creates a 2^{++} meson:

$$\Phi_{ij} = \bar{\psi}\left(\gamma_i D_j + \gamma_j D_i - \frac{2}{3}\delta_{ij}\gamma \cdot D\right)\psi$$

- Lattice: Substitute gauge-covariant lattice finite-difference D_{latt} for D
- A reducible representation:

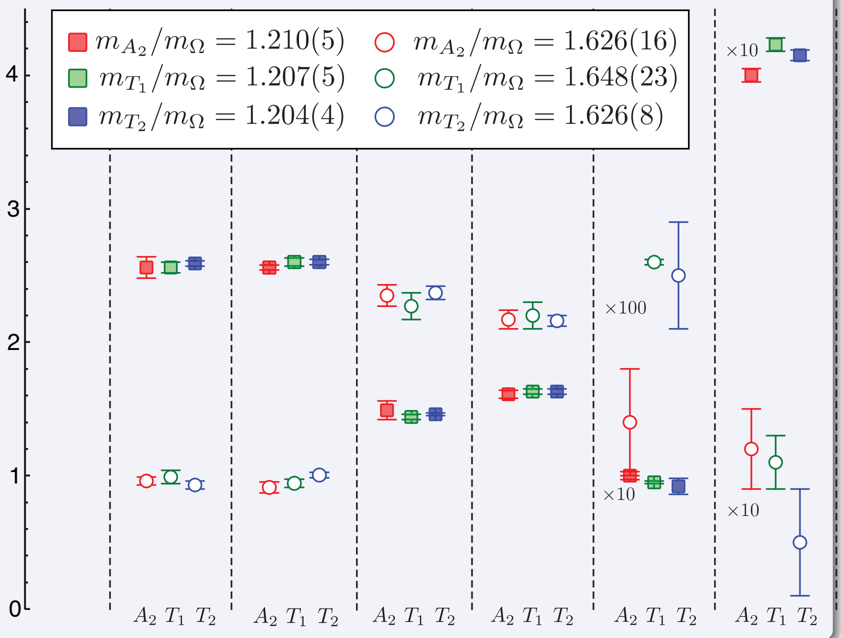
$$\Phi^{T_2} = \{\Phi_{12}, \Phi_{23}, \Phi_{31}\}$$

$$\Phi^{E} = \left\{\frac{1}{\sqrt{2}}(\Phi_{11} - \Phi_{22}), \frac{1}{\sqrt{6}}(\Phi_{11} + \Phi_{22} - 2\Phi_{33})\right\}$$

- Look for signature of continuum symmetry:

$$\langle 0|\Phi^{(T_2)}|2^{++(T_2)}\rangle = \langle 0|\Phi^{(E)}|2^{++(E)}\rangle$$

Spin-3 identification: J. Dudek *et.al.*, Hadron Spectrum Collab.



Bad news - the price tag

- So far - good results on modest lattice sizes $N_s = 16^3 \equiv (1.9\mathrm{fm})^3$.
- Used $N_{\mathcal{D}} = 64$ for mesons, $N_{\mathcal{D}} = 32$ for baryons

The problem:

- To maintain constant resolution, need $N_{\mathcal{D}} \propto N_s$
- **Budget:**

Fermion solutions	construct τ	$\mathcal{O}(N_s^{2})$
Operator constructions	construct Φ	$\mathcal{O}(N_s^{2})$
Meson contractions	$\mathrm{Tr}[\Phi\tau\Phi\tau]$	$\mathcal{O}(N_s^{3})$
Baryon contractions	$\bar{B}\tau\tau\tau B$	$\mathcal{O}(N_s^{4})$

- 32^3 lattice: $64 \times (\frac{32}{16})^3 = 512$ — too expensive.
- Some benefits in reduction in variance with N_s
- **Can stochastic estimation technology help?**

Stochastic estimation in the distillation space

- Construct a **stochastic identity matrix in $\mathcal{D}$**: introduce a vector η with $N_{\mathcal{D}}$ elements and with

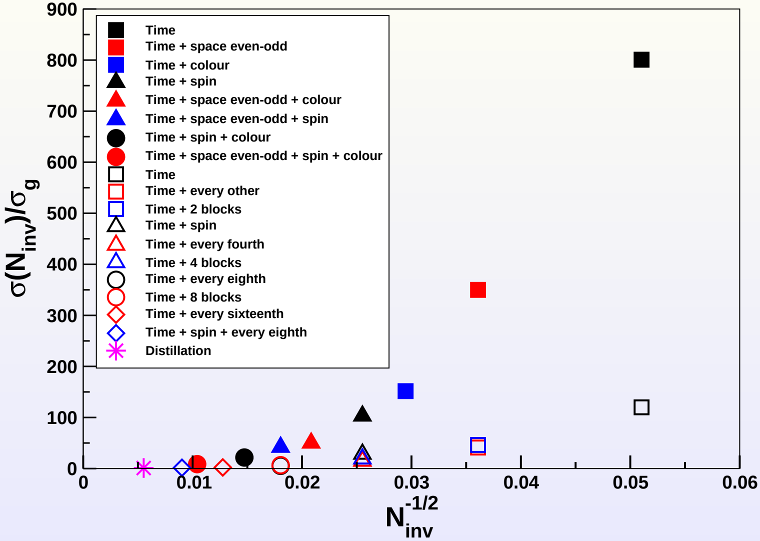
$$E[\eta_i] = 0 \text{ and } E[\eta_i \eta_j^*] = \delta_{ij}$$

- Now the distillation operator is written

$$\square = E[V\eta\eta^\dagger V^\dagger] = E[WW^\dagger]$$

- Introduces noise into computations
- **Dilution:** "thin out" the stochastic noise with N_η orthogonal projectors to make a variance-reduced estimator of $I_{\mathcal{D}} = E[WW^\dagger] = \sum_{k=1}^{N_\eta} E[V\mathcal{P}_k\eta\eta^\dagger\mathcal{P}_k V^\dagger]$, with $W_k = V\mathcal{P}_k\eta$, a $N_\eta \times (N_s \times N_c)$ matrix

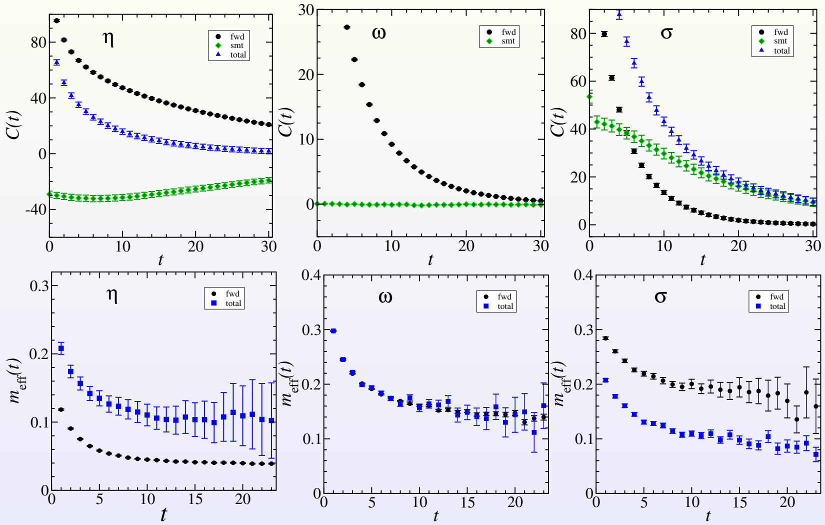
Stochastic estimation: baryon correlator



- Convergence faster for noise in distillation space

[arXiv:1011.0481]

Stochastic estimation: $I = 1, 0$ mesons

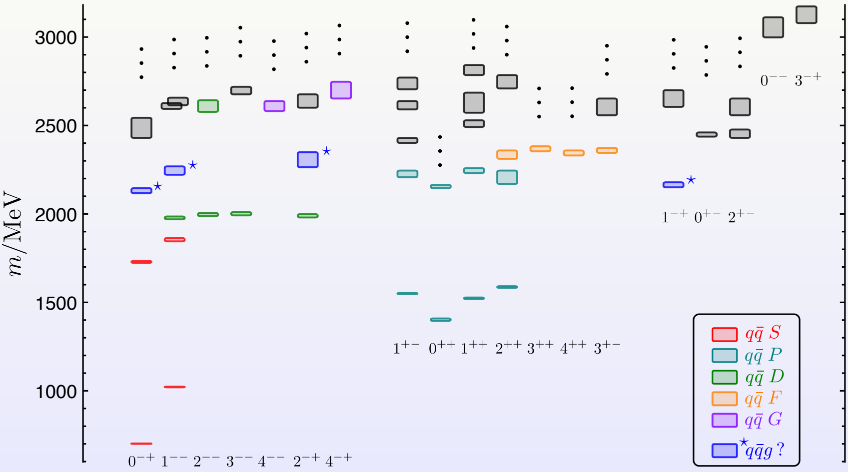


- propagation from $t - t$ is estimated differently from $t - t'$

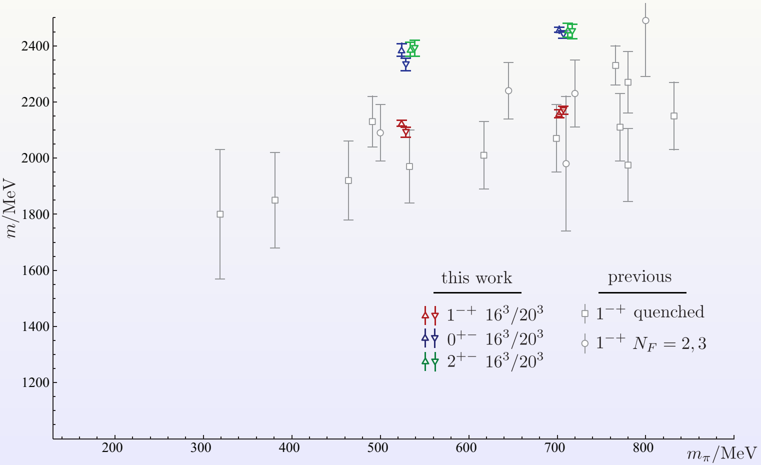
[arXiv:1101.5398v1]

Results: light hadrons

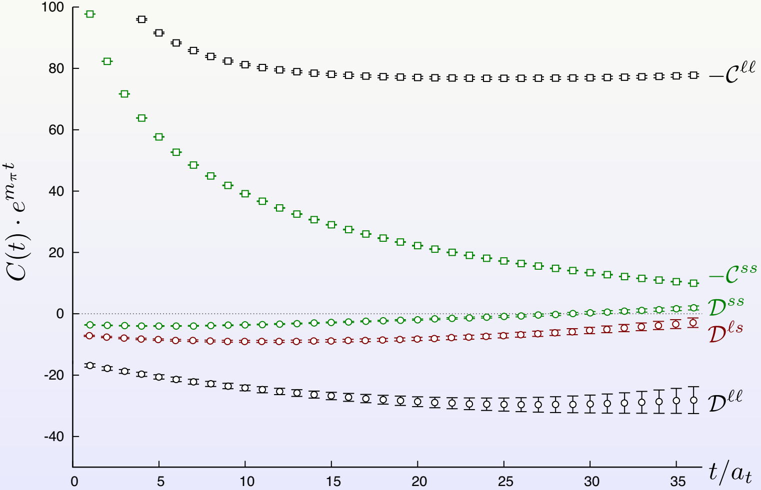
Isovector meson spectrum ($m_\pi = 702\text{MeV}$)



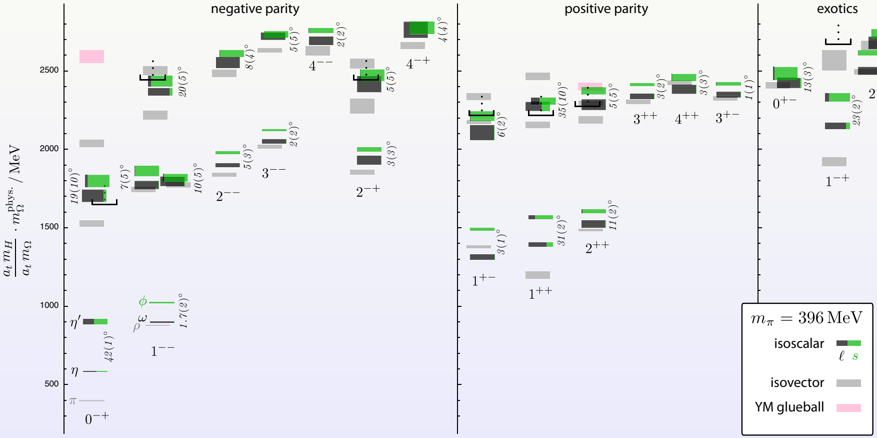
Exotic mesons



Isoscalar correlation functions



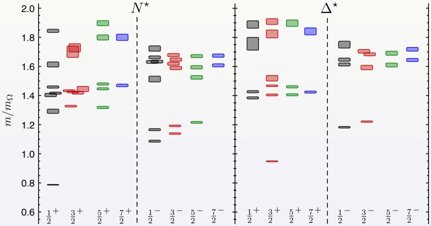
Isoscalar meson spectrum



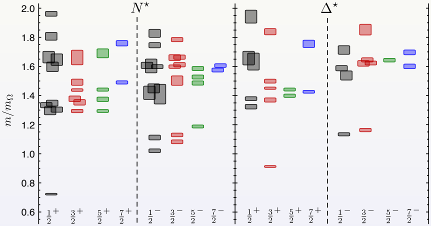
N and Δ excitations

[Edwards *et.al.*: arXiv:1104.5152]

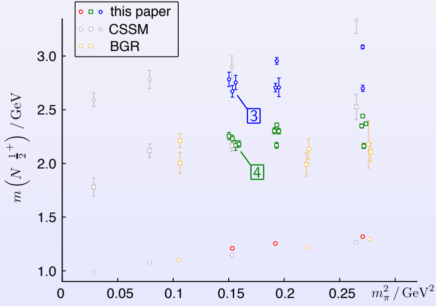
$m_\pi = 524$MeV



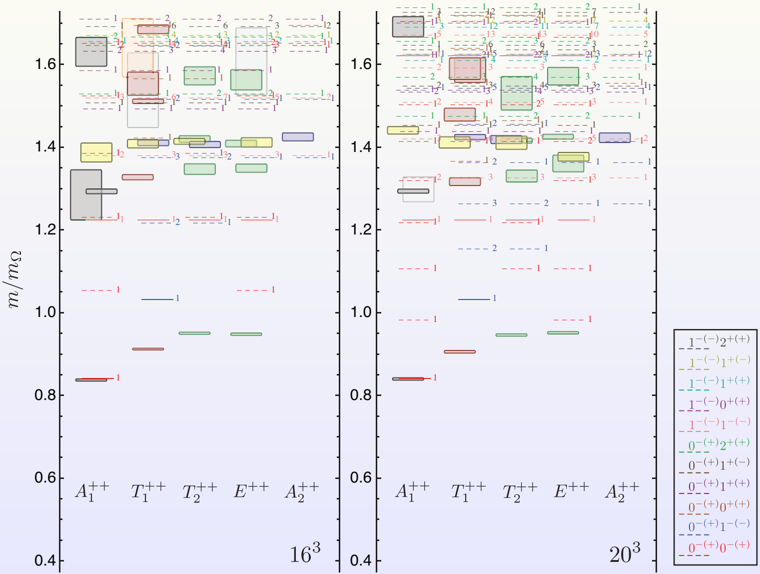
$m_\pi = 396$MeV



- Large operator basis, inspired by quark model
- With bigger operator basis, new states emerge
- More data closer to physical m_π required to understand the Roper

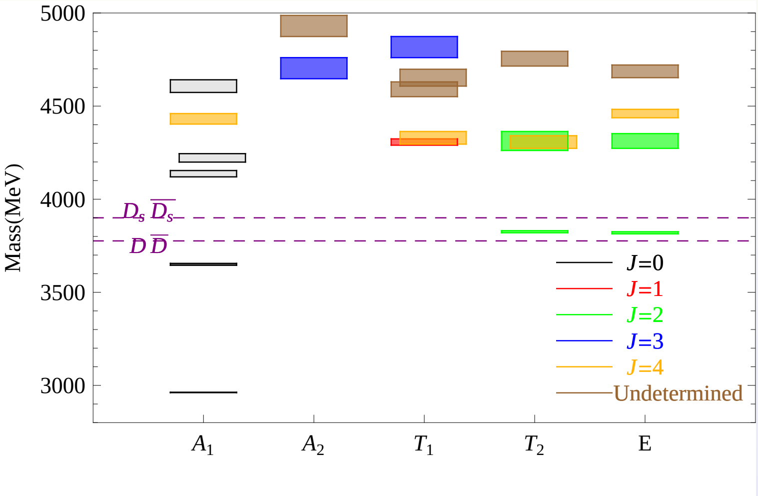


Where are the two-hadron states?

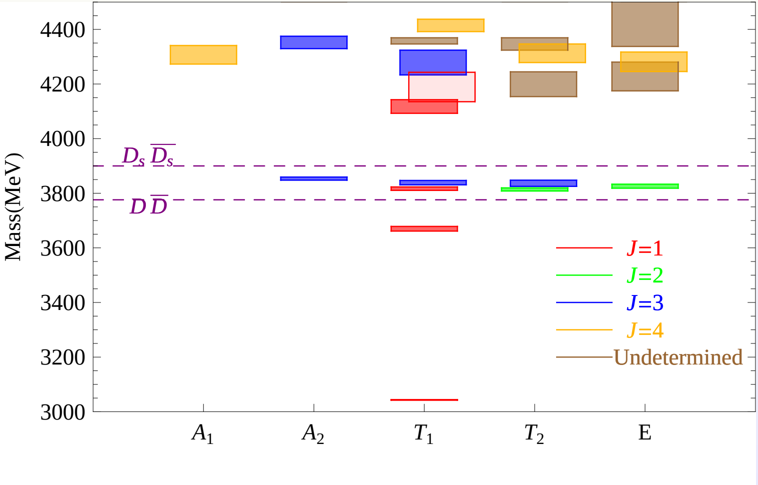


Charmonium

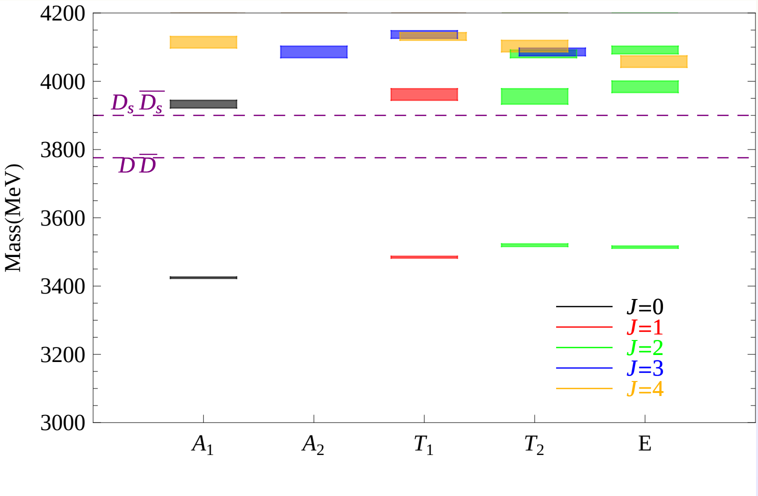
Charmonium: J^{-+}



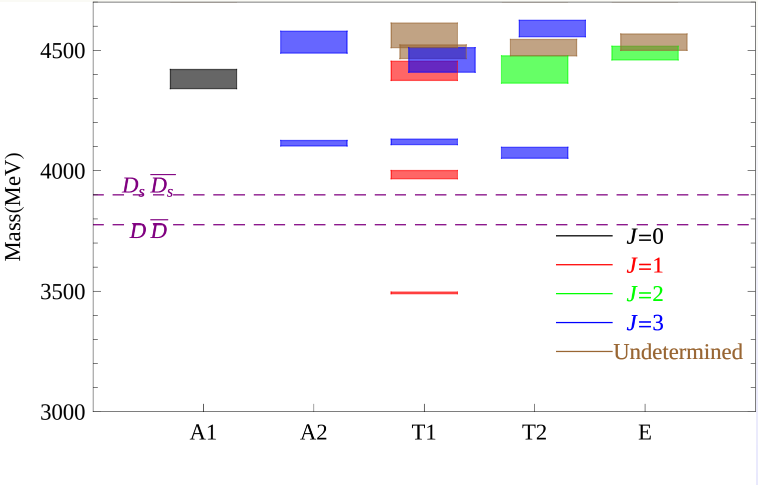
Charmonium: J^{--}



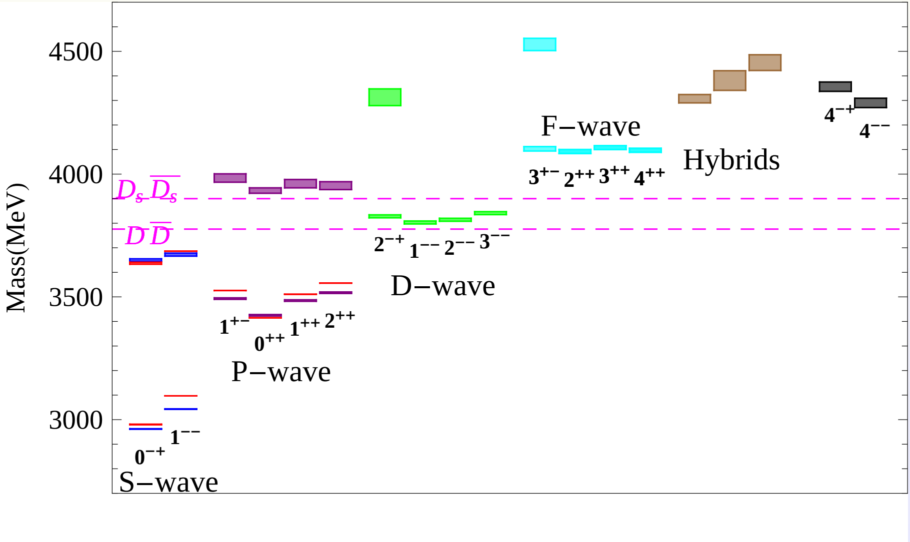
Charmonium: J^{++}



Charmonium: J^{+-}



Charmonium summary



Scattering and resonances

Particle(s) in a box

- Spatial lattice of extent L with periodic boundary conditions
- Allowed momenta are quantized: $p = \frac{2\pi}{L}(n_x, n_y, n_z)$ with $n_i \in \{0, 1, 2, \ldots L-1\}$
- Energy spectrum is a set of **discrete** levels, classified by p: Allowed energies of a particle of mass m

$$E = \sqrt{m^2 + \left(\frac{2\pi}{L}\right)^2 N^2} \quad \text{with } N^2 = n_x^2 + n_y^2 + n_z^2$$

- Can make states with **zero total momentum** from pairs of hadrons with momenta $p, -p$.
- “Density of states” **increases** with energy since there are more ways to make a particular value of N^2 e.g. $\{3, 0, 0\}$ and $\{2, 2, 1\} \rightarrow N^2 = 9$

Avoided level crossings

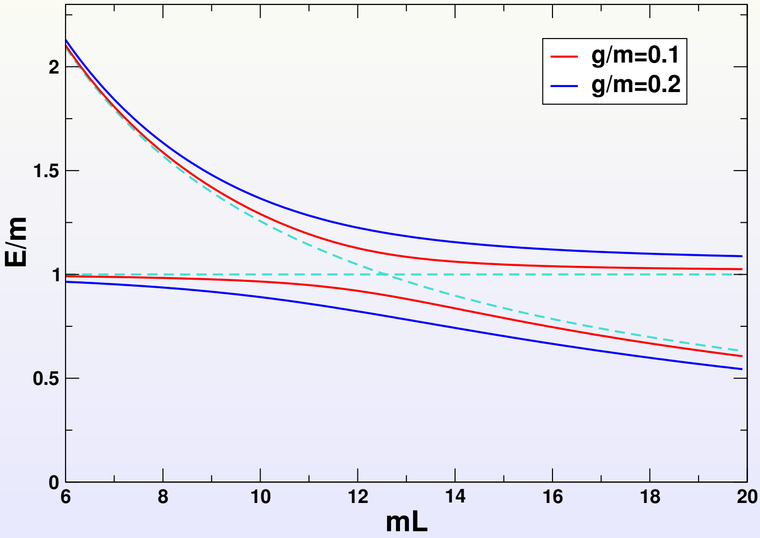
- Consider a toy model with two states (a resonance and a two-particle decay mode) in a box of side-length L
- Write a mixing hamiltonian:

$$H = \begin{pmatrix} m & g \\ g & \frac{4\pi}{L} \end{pmatrix}$$

- Now the energy eigenvalues of this hamiltonian are given by

$$E_{\pm} = \frac{(m + \frac{4\pi}{L}) \pm \sqrt{(m - \frac{4\pi}{L})^2 + 4g^2}}{2}$$

Avoided level crossings



Avoided level crossings

- **Ground-state** smoothly changes from resonance to two-particle state
- Need a large box. This example, levels cross at $mL = 4\pi \approx 12.6$
- Example: $m = 1$ GeV state, decaying to two massless pions - avoided level crossing is at $L = 2.5\text{fm}$.
- If the decay product pions have $m_\pi = 300$ MeV, this increases to $L = 3.1\text{fm}$

Lüscher's method

- Relates the spectrum in a finite box to the scattering phase shift (and so resonance properties)

Lüscher's formula

$$\delta(p) = -\phi(\kappa) + \pi n$$

$$\tan\phi(\kappa) = \frac{\pi^{3/2}\kappa}{Z_{00}(1;\kappa^2)}$$

$$\kappa = \frac{pL}{2\pi}$$

- p_n is defined for level n with energy E_n from the dispersion relation:

$$E_n = 2\sqrt{m^2 + p_n^2}$$

Lüscher's method

- Z_{00} is a generalised Zeta function:

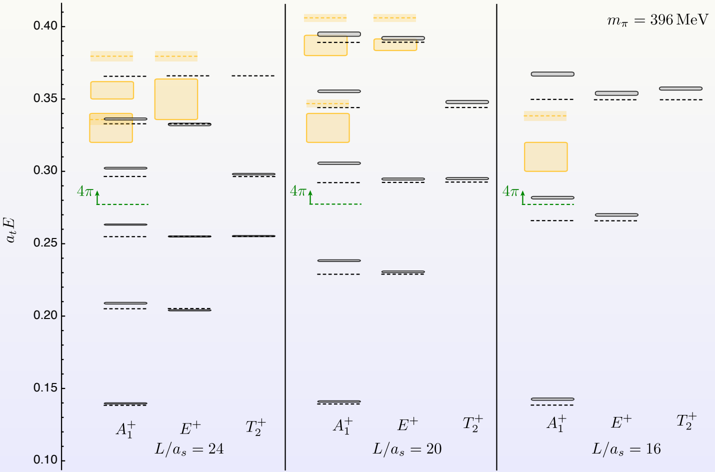
$$Z_{js}(1, q^2) = \sum_{n \in Z^3} \frac{r^j Y_{js}(\theta, \phi)}{(n^2 - q^2)^s}$$

[M.Lüscher, Commun.Math.Phys.105:153-188,1986.]

- With the phase shift, and for a well-defined resonance, can fit a Breit-Wigner to extract the **resonance width** and **mass**.

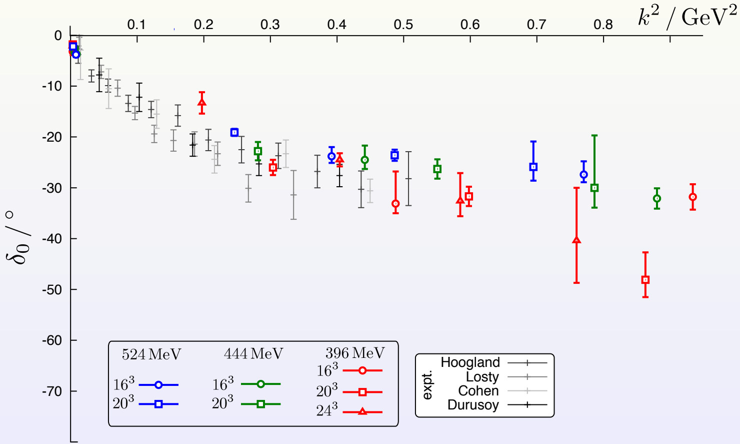
$$\delta(p) \approx \tan^{-1}\left(\frac{4p^2 + 4m_\pi^2 - m_\sigma^2}{m_\sigma \Gamma \sigma}\right)$$

I=2 $\pi\pi$ scattering



Resolve shifts in masses away from non-interacting values

I=2 $\pi\pi$ scattering



- Non-resonant scattering in S-wave - compares well with experimental data

Conclusions

- More sophisticated tools are needed for precision:
  - exotic and excited state spectroscopy
  - isoscalar spectroscopy
  - calculations with multi-hadron states
- **Distillation** provides a useful framework to develop better tools
- Good first results for
  - excited mesons (I=1 and I=0)
  - excited baryons
  - charmonium
  - mesons in flight
- **Bad news:** the price tag. Poor volume scaling
- Solutions include finding better distillation spaces and using good stochastic estimation schemes.