Explicit and spontaneous breaking of SU(3) into its finite subgroups

Roman Zwicky
University of Southampton

based on arXiV.1100.4891v1 in collaboration Alexander Merle (Stockholm)
*along Mathematica package **SUtree** lots of features*
http://theophys.kth.se/~amerle/SUtree/SUtree.html

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0. Prologue -- finite groups

not required for talk (might want to hear anyway)

☆ Finite group F = group finite number of elements

- Example: *three permutations* **S₃**: Order $|S_3|=3!=6$ ( ),(12),(23),(13),(123),(132)

☆ Dimensionality thm: $\boxed{|F| = \sum |\mathrm{IRREPS}(F)|^2}$ ⇒ finite many of them

- $|S_3| = |\mathbf{1}|^2 + |\mathbf{1'}|^2 + |\mathbf{2}|^2$

☆ Characterized by charactertable (Brauer hypothesis ....)

- conjugacy classes $c' \sim gcg^{-1}$ $g \in F$
  character $\chi = \mathrm{tr}[c]$

S_3	$\{e\}$	$\{a,b,c\}$	$\{d,f\}$
Γ_1	1	1	1
Γ_2	1	α	β
Γ_3	2	γ	δ

- Almost all information e.g. Kronecker products $(\mathbf{2} \times \mathbf{2})_{S3} = (\mathbf{1}+\mathbf{1}+\mathbf{2})_{S3}$

☆ Many others than permutation groups -- any finite group can be embedded in a permutation group

Outline

- I. Introduction

- II. Main ideas -- using SO(3) → S_4 as an example
  - II.a Classification of invariants
  - II.b Exemplified S_4
  - II.c Problem of sufficient condition

- III. Finite subgroups of SU(3) -- denoted by F_3

- IV. Database

- V. Example criteria for breaknig SU(3) → F_3

- VI. Complex spherical harmonics

- VI. Tensor-generating function (generalization of Molien function)

- Epilogue (talk about open ends)

1. Introduction

Groups in "disguises"

consider S_4: the group of permutation of four objects

- geometric definition through irreps: $4! = |S_4| = \mathbf{1}^2 + \mathbf{1}'^2 + \mathbf{3}^2 + \mathbf{3}'^2$ (dim thm)
  (to each of four objects assign orthogonal vectors -- implement permutation linearly)
- abstract algebraic definition $S_4 = \langle\langle$ words a,b, $| \; a^3=1, b^2=1, (ab)^4=1 \rangle\rangle$

today's particle physicist more familiar with Lie groups e.g. O(3)

- $O(3) = \langle\langle M_3 \mid M_3^T M_3 = 1 \rangle\rangle$

equivalently all linear operations in three variables that leave $x^2 + y^2 + z^2$ invariant.

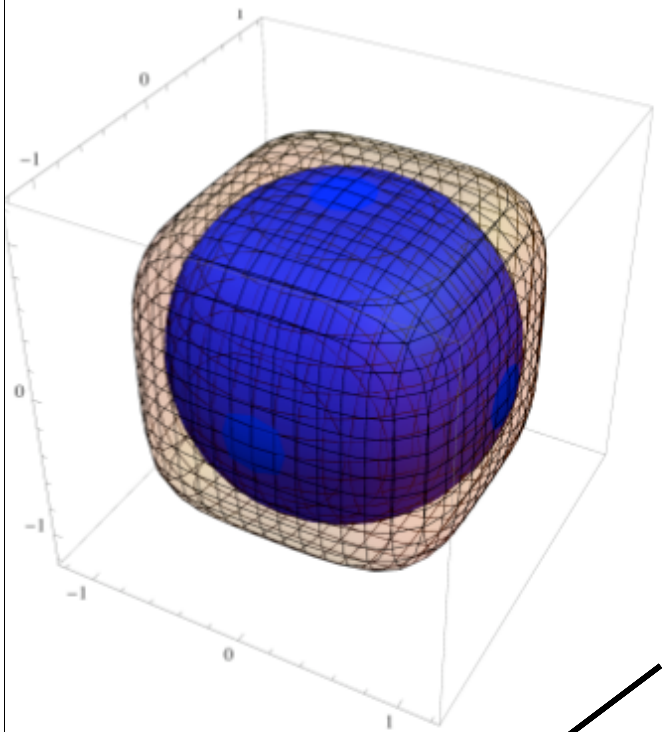
is there an analogous way to think about finite groups?

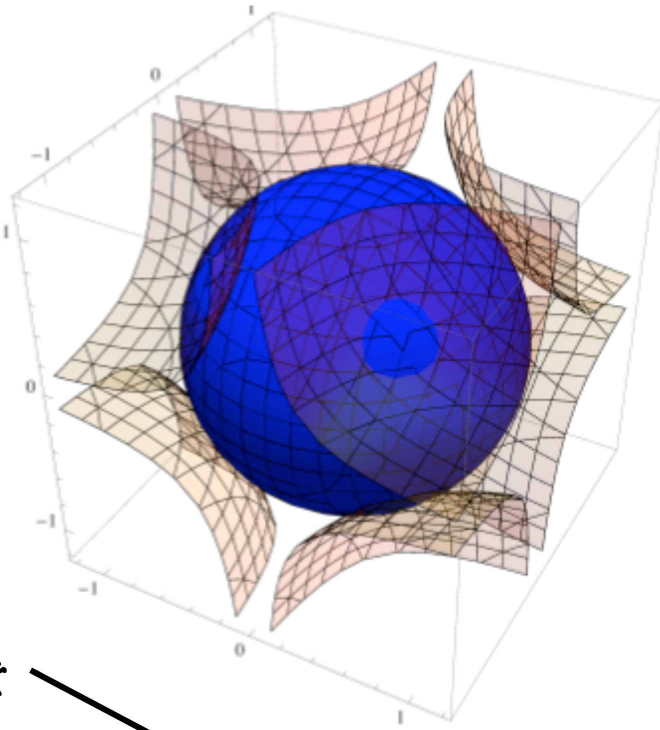
Yes: visual “proof” for SO(3) → S_4

S_4 isomorphic cubic-symmetry
(dual octahedron)

$x^4 + y^4 + z^4$

$(xyz)^2$





intersections of sphere SO(3)
and add. invariant = constant

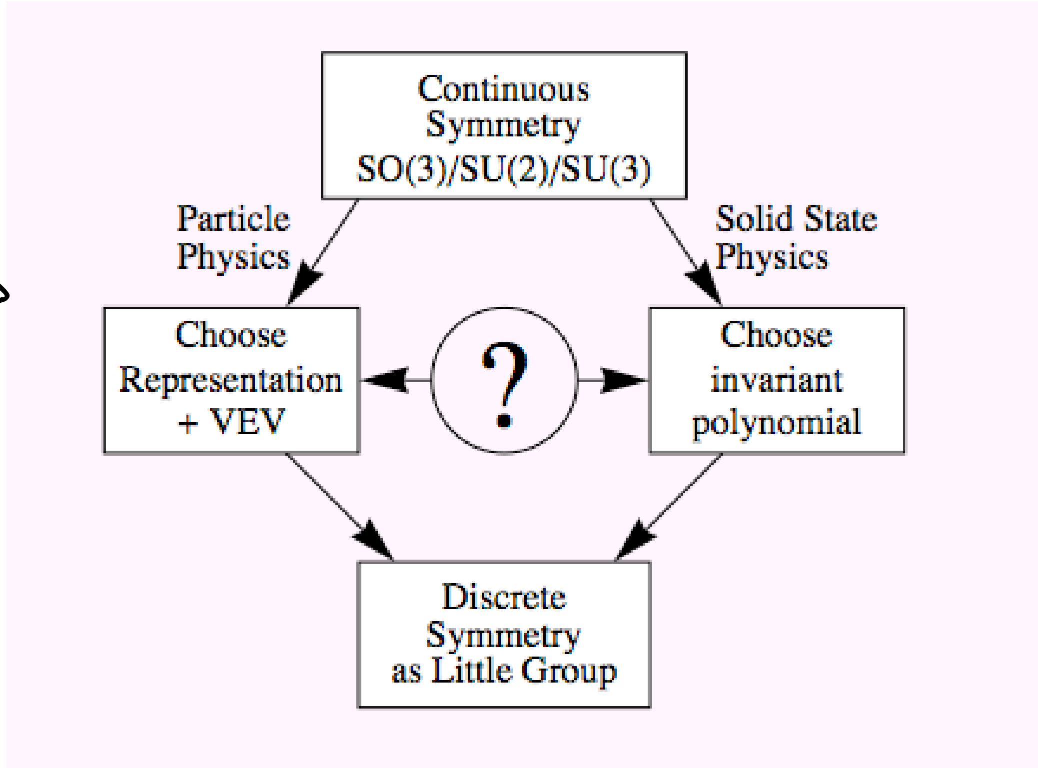
vertices of octahedron

vertices of cube

Turning to the physicist's vocabulary

Definition of a group ⇔ Conditions breaking into this group

Spontaneous
symmetry breaking



Explicit
symmetry breaking

➥ This talk: - link between SSB & explicit breaking
- classification invariants,
- necessary conditions to define group

for SU(3)

II. Main ideas ...

... discussed within $SO(3) \rightarrow S_4$

Spontaneous symmetry breaking (through vacuum)

☆ Consider fundamental irrep SO(3) i.e. $\mathbf{3}_{(l=1)}$

vacuum vector $\uparrow$ $\Rightarrow$ SO(2) symmetry remains ; not enough

☆ Consider higher dimensional irrep l = 2,3,4,5 ; abandon geometric picture

Proceeding abstract manner:
Choose vacuum v, then $H_v = \{g \in SO(3) \mid R(g)v = v\} \subset R(SO(3))$ defines a subgroup

➥ $SO(3) \to H$ through VEV v

Explicit symmetry breaking (through invariants)

☆ Consider $\varphi_i \in \mathbf{3}_{(l=1)}$ $L_{tot} = L_{SO(3)}(|\varphi|^2) + L_{break(H)}(\varphi_1, \varphi_2, \varphi_3)$

Again any $L_{break(H)}$ will break $SO(3) \to H$ some subgroup

From our geometric "proof" $L_{S4} = \varphi_1^{\ 4} + \varphi_2^{\ 4} + \varphi_3^{\ 4}$

One-to-one link between spontaneous & explicit symmetry breaking

- How can polynomials be linked to vector spaces?
  ⇒ Groups have polynomial representation functions

spherical harmonics $Y_{l,m}$
for SO(3)
= complete set of irreps

$$\mathcal{I}[S_4] = x^4 + y^4 + z^4 = c\left(Y_{4,-4} + \sqrt{\frac{14}{5}}Y_{4,0} + Y_{4,4}\right)$$

- This means that choosing a VEV:

$$v \sim (1,0,0,0,\sqrt{\tfrac{14}{5}},0,0,0,1) \quad \text{then} \quad \mathbf{9}_{l=4}|_{S_4} \to \mathbf{1}_{S_4} + \ldots$$

branching rule

  which is easily verified explicitly

- Extension to SU(3) involves finding SU(3) representation function
  ⇒ complex spherical harmonics (studied in 60's) (discuss latter)

II.a Classification of invariants

- all polynomial invariants } general
- algebraic dependencies etc

Molien's theorem (1897)

$$M_{\mathcal{R}(H)}(P) \equiv \frac{1}{|\mathcal{R}(H)|} \sum_{h \in \mathcal{R}(H)} \frac{1}{\det(\mathbf{1} - P\,h)} = \sum_{m \geq 0} h_m P^m \ ,$$

easy to compute

R(H) is an irrep of a finite group H.
Thm: Positive coefficients h_m count the number of polynomial invariants of degree m.

Algebraic dependence

- For n variables there are n algebraically independent invariants (Noether 1916)
  Those we call primary and all the other secondary invariants.

- Dependence of secondary invariants as follows:

$$\overline{\mathcal{I}}_{n_i}^{\,2} = f_0(\mathcal{I}_{m_1}, \mathcal{I}_{m_2}, \mathcal{I}_{m_3}) + \sum_j f_1^{(j)}(\mathcal{I}_{m_1}, \mathcal{I}_{m_2}, \mathcal{I}_{m_3}) \cdot \overline{\mathcal{I}}_{n_j} \ ,$$

syzygy

☆ Fact: If degrees primary & secondary invariants known then the Molien fct assumes ..

$$\{\mathcal{I}_{m_1}, \mathcal{I}_{m_2}, \mathcal{I}_{m_3}, \overline{\mathcal{I}}_{n_i}, ..\} \quad \Rightarrow \quad M_{H(\mathbf{3})}(P) = \frac{1 + \sum_i a_{n_i} P^{n_i}}{(1 - P^{m_1})(1 - P^{m_2})(1 - P^{m_3})} \; .$$

The form of the Molien fct is <u>not unambiguous</u> thus no ⇐ implication

☆ ⇒ establishing primary & secondary invariants is non-trivial

In practice: 1) guess form of Molien fct as above*
2) generate invariants
3) verify syzygies (great sport)

Thm: $$\text{number of secondary invariants} \equiv 1 + \sum_i a_{n_i} = \frac{m_1 \cdot m_2 \cdot m_3}{|H|} \; ,$$

☆ Generating invariants: symmetrize over group (Reynold operator)

$$\mathcal{I}(x, y, z) = \frac{1}{|\mathcal{R}(H)|} \sum_{h \in \mathcal{R}(H)} f(h \circ x, h \circ y, h \circ z) \; ,$$

for any ansatz f(x,y,z), I is an invariant

II.b Exemplified with S$_4$

- Molien function takes form (level of ambiguity low)

$$M_{S_4}(P) = \frac{1+P^9}{(1-P^2)(1-P^4)(1-P^6)} \ ,$$

- The following (candidate) primary and secondary invariants are found

$$\mathcal{I}_2[S_4] = x^2+y^2+z^2 \ , \quad \mathcal{I}_6[S_4] = (xyz)^2 \ , \quad \mathcal{I}_4[S_4] = x^4+y^4+z^4 \ ,$$

$$\overline{\mathcal{I}}_9[S_4] = xyz(x^2-y^2)(y^2-z^2)(z^2-x^2) \ ,$$

- The one and only syzygy is:

$$\overline{\mathcal{I}}_9^2 = \mathcal{I}_2^4\mathcal{I}_4\mathcal{I}_6 - \frac{1}{4}\mathcal{I}_2^6\mathcal{I}_6 - \frac{5}{4}\mathcal{I}_2^2\mathcal{I}_4^2\mathcal{I}_6 + \frac{1}{2}\mathcal{I}_4^3\mathcal{I}_6 + 5\mathcal{I}_2^4\mathcal{I}_6^2 - 9\mathcal{I}_2\mathcal{I}_4\mathcal{I}_6^2 - 27\mathcal{I}_6^3 \ ,$$

II.c Problem of sufficient criteria for breaking G → H

- Is the hard problem (in the sense that there's no general stratetgy)
  SO(3) famous Michel criterion '79 counterexamples found

- Illustration of the problem:
  Fact 1: A_4, S_4 both leave $I_4 = x^4 + y^4 + z^4$ invariant
  Fact 2: A_4 is a subgroup of S_4
  $\Rightarrow$ I_4 breaks SO(3) into S_4 (if at all) but <u>not</u> into A_4

- $\Rightarrow$ imposing I_X, SO(3) breaks into maximal subgroup for which I_X is an invariant.

- ought to know entire subgroup tree from G to H (and their invariants)
  not known in general .....
  finding subgroups of say SU(n) seems case by case study -- more thought later

no general strategy → look example

III. Finite subgroups of SU(3)

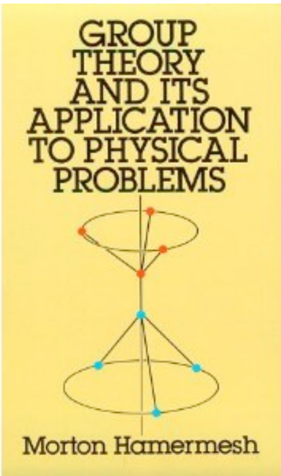
- Of interest flavour model building
- Alternatives to $SU(3)_F$ (eighfold way)
- Discretization of $SU(3)_c$ for lattice e.g. Michael et al

Finite subgroups of SU(3) denoted by F_3

- Classified in a classic book Miller, Dickson, Blichfeld '1916 , analyzed further 8-fold way Fairbairn, Fulton, Klink '64 Further analyzed (lattice ...) Bovier, Luling, Wyler '80 Rescrutinized tri-bi-hype Luhn, Nasri Ramand; Fischbacher RZ, Ludl, Grimus 03' onwards

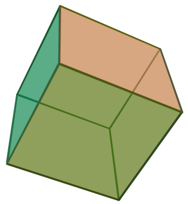
- First SO(3) subgroups (3d irreps) -- then algebraic abstraction SU(3)

Dihedral $D_n = Z_n \rtimes Z_2$



- symmetries of a molecule
- irreps smaller equal to 3

Crystallographic



cube=S_4

tetrahedron=A_4

icosahedron=A_5

"better" approximation to SO(3)

Algebraic abstraction to SU(3) (not simple factor groups Z_3 x ....)

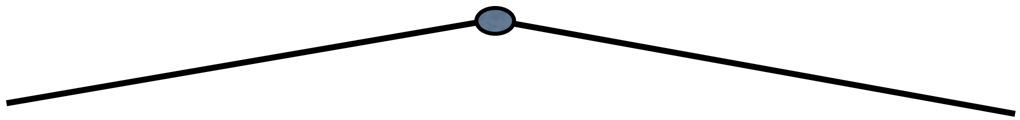
C,D-groups
= dihedral-like
= trihedral

crystallographic type

Group	Generators C-,D-type	$\Sigma(X)$-type
$C(n,a,b)$	$E,\ F(n,a,b)$	
$D(n,a,b;d,r,s)$	$E,\ F(n,a,b),\ G(d,r,s)$	
$\Delta(3n^2) = C(n,0,1),\ n \geq 2$	$E,\ F(n,0,1)$	
$\Delta(6n^2) = D(n,0,1;2,1,1),\ n \geq 2$	$E,\ F(n,0,1),\ G(2,1,1)$	
$T_{n[a]} = C(n,1,a),\ (1+a+a^2) = n\mathbb{Z}$	$E,\ F(n,1,a)$	
$\Sigma(60) = A_5 = I = Y$	$E,\ F(2,0,1)$	H
$\Sigma(168) = PSL(2,7)$	$E,\ M \equiv F(7,1,2)$	N
$\Sigma(36\phi)$	$E,\ J \equiv F(3,0,1)$	K
$\Sigma(72\phi)$	$E,\ J \equiv F(3,0,1)$	$K,\ L$
$\Sigma(216\phi)$	$E,\ J = F(3,0,1),\ P \equiv F(9,2,2)$	K
$\Sigma(360\phi)$	$E,\ F(2,0,1),\ Q \equiv G(6,3,5)$	H

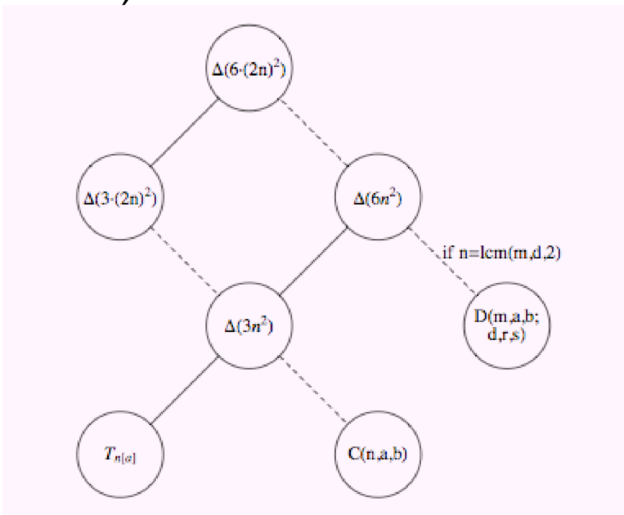
known generators

- general C,D-groups not everything is known (continuous progress ...)
  known: topological structure irreps smaller 3/6 respectively
  unknown: order group subgroup (partial results)



General

Database....

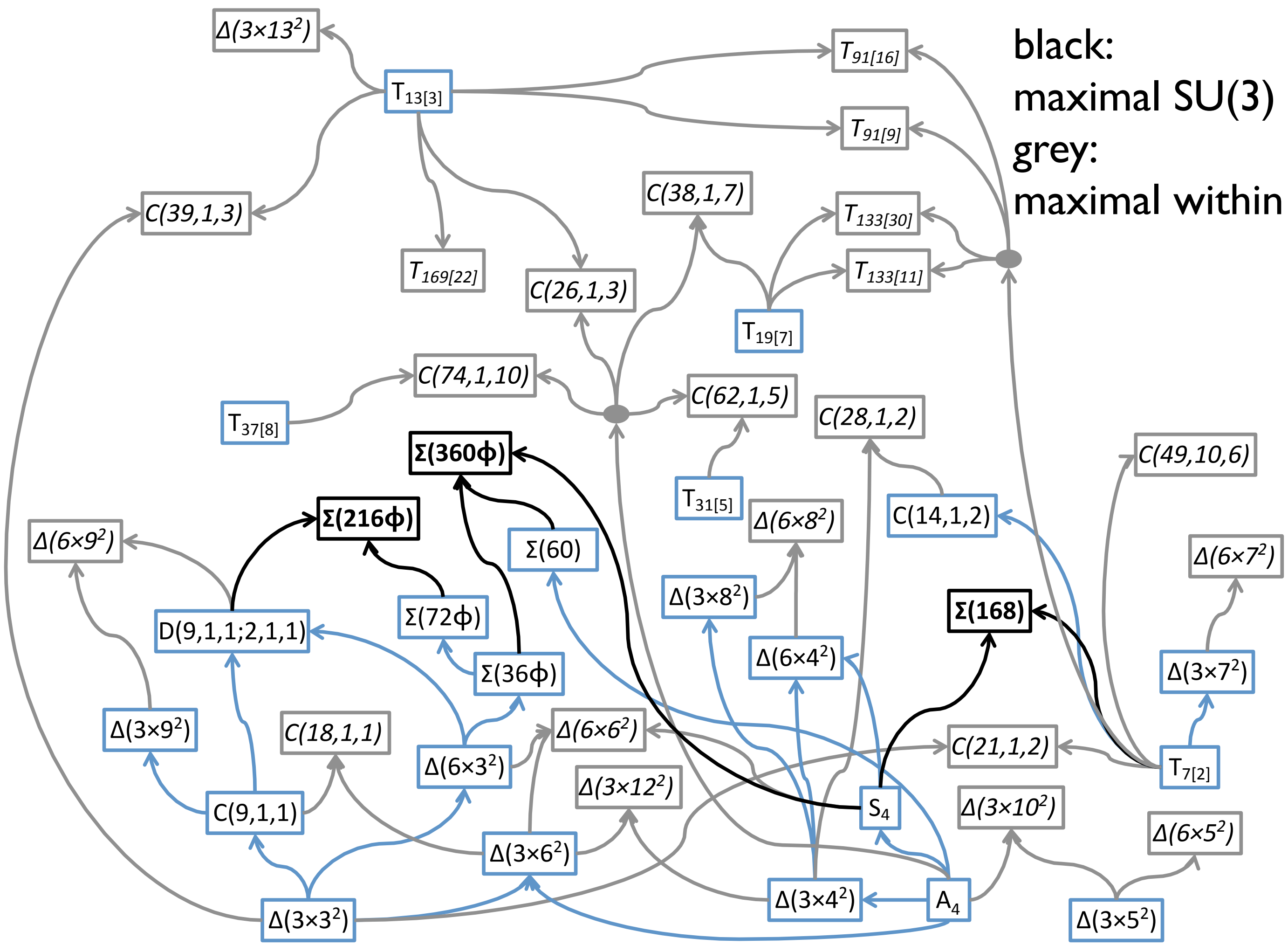


work out sufficient criteria
SU(3) → F_3 (later)

IV. Database: groups of order smaller 512 (61 of them)

- Find all syzygies and thus primary & secondary invariants, Molien function, tensor-generating functions.. Finding syzygies is an interesting problem complexity (use polynomial basis ...) Especially crystallographic ones of interest for mathematicians

- and here's the tree:



black:
maximal SU(3) subgroups
grey:
maximal within database

V. Example criteria for breaking $SU(3) \rightarrow F_3$

☆ SU(3) → F_3 ..... problem to know $F_3 \subset H \subset SO(3)$

- H might be continuous SO(3) and SU(2) and subgroups thereof
  a) SO(3) know all the subgroups ok
  b) notice all subgroups have cyclic generator E (x,y,z)→(y,z,x)   SU(2) out of the game
- H mixed ... out for the same reason
- H finite one of our list ⇒ work with explicit generators

justify
as generators
specific embedding

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad F(n,a,b) = \begin{pmatrix} \eta^a & 0 & 0 \\ 0 & \eta^b & 0 \\ 0 & 0 & \eta^{-a-b} \end{pmatrix}, \quad G(d,r,s) = \begin{pmatrix} \delta^r & 0 & 0 \\ 0 & 0 & \delta^s \\ 0 & -\delta^{-r-s} & 0 \end{pmatrix},$$

$$H = \frac{1}{2}\begin{pmatrix} -1 & \mu_- & \mu_+ \\ \mu_- & \mu_+ & -1 \\ \mu_+ & -1 & \mu_- \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad K = \frac{1}{\sqrt{3}\,i}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

$$L = \frac{1}{\sqrt{3}\,i}\begin{pmatrix} 1 & 1 & \omega^2 \\ 1 & \omega & \omega \\ \omega & 1 & \omega \end{pmatrix}, \quad M = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \beta^4 \end{pmatrix}, \quad N = \frac{i}{\sqrt{7}}\begin{pmatrix} \beta^4-\beta^3 & \beta^2-\beta^5 & \beta-\beta^6 \\ \beta^2-\beta^5 & \beta-\beta^6 & \beta^4-\beta^3 \\ \beta-\beta^6 & \beta^4-\beta^3 & \beta^2-\beta^5 \end{pmatrix},$$

$$P = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon\omega \end{pmatrix}, \quad Q = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & -\omega^2 & 0 \end{pmatrix}. \qquad \text{(B.1)}$$

$$\eta \equiv e^{2\pi i/n}, \quad \delta \equiv e^{2\pi i/d}, \quad \mu_\pm \equiv \frac{1}{2}\left(-1 \pm \sqrt{5}\right), \quad \omega \equiv e^{2\pi i/3}, \quad \beta \equiv e^{2\pi i/7}, \quad \epsilon \equiv e^{4\pi i/9}.$$

V.a Crystallographic groups

- Module embedding rather straighforward (just apply all generators to them.._

Group	Molien function	Invariant of lowest degree that breaks $SU(3) \to \Sigma(X)$
$\Sigma(60)$	$\frac{1+P^{15}}{(1-P^2)(1-P^6)(1-P^{10})}$	$(\phi_0^2 x^2 - y^2)(\phi_0^2 z^2 - x^2)(\phi_0^2 y^2 - z^2)$
$\Sigma(36\phi)$	$\frac{1+P^9+P^{12}+P^{21}}{(1-P^6)^2(1-P^{12})}$	$(x^6 + 2x^3y^3 - 6x^4yz + \text{cy.}) - 18x^2y^2z^2$
$\Sigma(168)$	$\frac{1+P^{21}}{(1-P^4)(1-P^6)(1-P^{14})}$	$x^3z + z^3y + y^3x$
$\Sigma(72\phi)$	$\frac{1+P^{12}+P^{24}}{(1-P^6)(1-P^9)(1-P^{12})}$	$x^6 + y^6 + z^6 - 10x^3y^3 - 10y^3z^3 - 10z^3x^3$
$\Sigma(216\phi)$	$\frac{1+P^{18}+P^{36}}{(1-P^9)(1-P^{12})(1-P^{18})}$	$x^6(y^3 - z^3) + y^6(z^3 - x^3) + z^6(x^3 - y^3)$
$\Sigma(360\phi)$	$\frac{1+P^{45}}{(1-P^6)(1-P^{12})(1-P^{30})}$	$x^6 + y^6 + z^6 + ax^2y^2z^2 + b_+\,(x^4y^2 + \text{cy.}) + b_-\,(x^4z^2 + \text{cy.})$

famous **Klein-quartic**

$$\phi_0 \equiv \frac{1+\sqrt{5}}{2}\ ,\ a = 3\left(5 - i\sqrt{15}\right)\ ,\ b_\pm = \frac{3}{8}\left[5 \mp 3\sqrt{5} + i\left(\sqrt{15} \pm 5\sqrt{3}\right)\right]$$

V.b $\Delta(6n^2)$, $\Delta(3n^2)$, $T_{n[a]}$-series

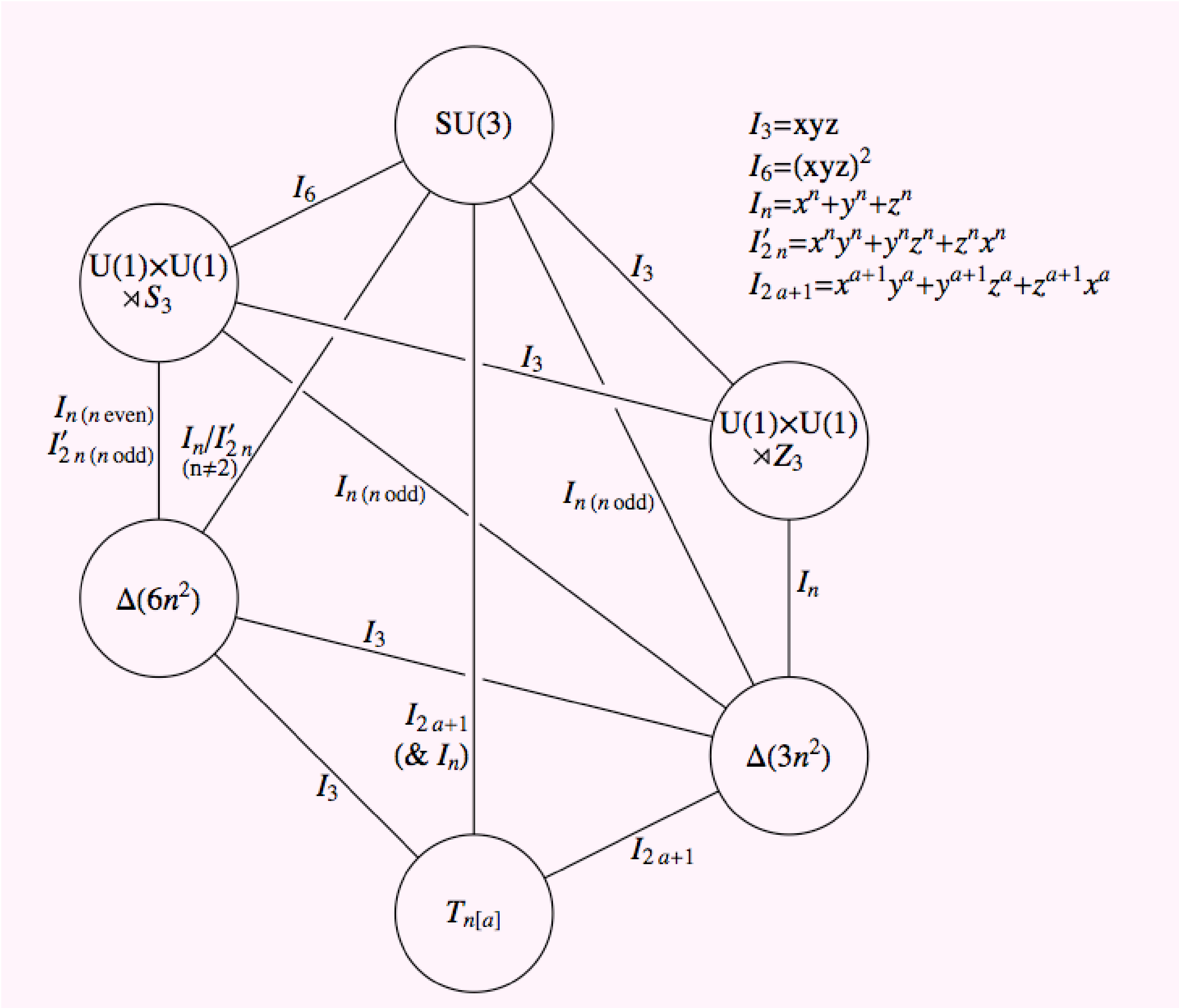
- are the known series amongst C/D subgroups -- dihedral-like
- insight comes from invariants & syzygies -- doable for general n (exceptional)!

Group	Type	Invariants
$\Delta(3n^2)$	primary	$\mathcal{I}_3 = xyz,$
		$\mathcal{I}_n = x^n + y^n + z^n$
		$\mathcal{I}_{2n} = x^{2n} + y^{2n} + z^{2n}$
	secondary	$\mathcal{I}_{3n} = x^{3n} + y^{3n} + z^{3n}$
	syzygy	$\overline{\mathcal{I}}_{3n}^2 = 9\mathcal{I}_3^{2n} + 9\mathcal{I}_3^n \mathcal{I}_n \mathcal{I}_{2n} + \frac{9}{4}\mathcal{I}_n^2 \mathcal{I}_{2n}^2 - 3\mathcal{I}_3^n \mathcal{I}_n^3 - \frac{3}{2}\mathcal{I}_n^4 \mathcal{I}_{2n} + \frac{1}{4}\mathcal{I}_n^6$
$\Delta(6n^2)$ even n	primary	$\mathcal{I}_6 = (xyz)^2,$
		$\mathcal{I}_n = x^n + y^n + z^n$
		$\mathcal{I}_{2n} = x^{2n} + y^{2n} + z^{2n}$
	secondary	$\overline{\mathcal{I}}_{3n+3} = xyz(x^n - y^n)(y^n - z^n)(z^n - x^n)$
	syzygy	$\overline{\mathcal{I}}_{3n+3}^2 = \mathcal{I}_6 \left[\frac{1}{2}\mathcal{I}_{2n}^3 - 27\mathcal{I}_6^n - 9\mathcal{I}_6^{n/2}\mathcal{I}_n \mathcal{I}_{2n} + 5\mathcal{I}_6^{n/2}\mathcal{I}_n^3 + \mathcal{I}_n^4 \mathcal{I}_{2n} - \frac{1}{4}\mathcal{I}_n^6 - \frac{5}{4}\mathcal{I}_n^2 \mathcal{I}_{2n}^2\right]$

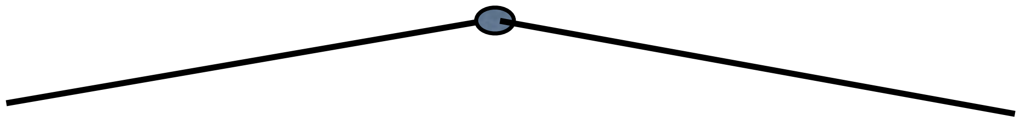
suggests the that the dihdral-like groups are generalizations
tetrahedron/cube by changing the euclidian metric

$$
\begin{array}{lclcl}
A_4; S_4 : & \rightarrow & \Delta(3n^2); \Delta(6n^2)_{n\in 2\mathbb{N}} : S_3 : & \rightarrow & \Delta(6n^2)_{n \in 2\mathbb{N}+1} : \\
x^2 + y^2 + z^2 & \rightarrow & x^n + y^n + z^n \,. \quad\quad xy + yz + zx & \rightarrow & x^n y^n + y^n z^n + z^n x^n \,,
\end{array}
$$

after tedious (yet fun -- geometric intuition) work we were able to show:



V.c Hint at questions of embedding



equivalent embedding
= similarity transformation

$g' = AgA^{-1}$
using Schur's Lemma
& subgroup tree show
not "lost" anything

inequivalent embedding
= distinct irrep

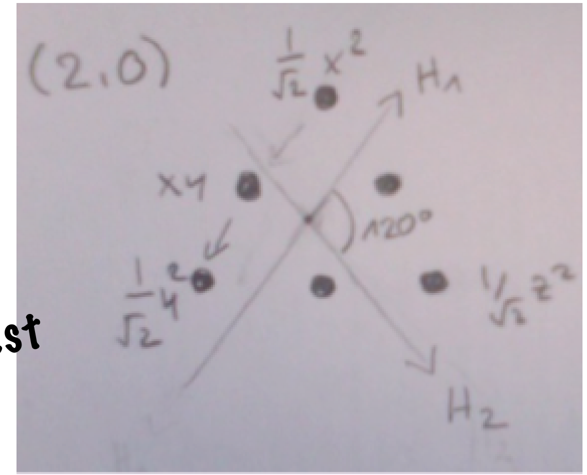
e.g. A_5 $\mathbf{3}, \mathbf{3}'$
show image same or complex conjugate
(latter case particle/anti-particle)

VI. The complex spherical harmonics

SU(3)-representation functions

- eigenfunctions of Laplacian on SU(3)/SU(2)

- formal construction SU(3) rank 2 -- basis Cartan subalgebra $\{T_3, T_8\}$
  $T_3 = \frac{1}{2}(x\partial_x - y\partial_y)\ , \quad T_8 = ..$ highest weight (p,0) $\leftrightarrow x^p$



- (p,q) = { polynom degree (p,q) in $(\{x,y,z\},\{x^*, y^*, z^*\})\}/\{xx^* + yy^* + zz^*\}$

simplest

- comparison of SO(3) vs SU(3)

group	$SO(3)$	$SU(3)$
rank	$1 \leftrightarrow l$	$2 \leftrightarrow (p,q)$
repres. fct.	$Y_{l,m}$	$h^{rst}_{(p,q)}$
fct. on manifold	$SO(3)/SO(2) \simeq S_2$	$SU(3)/SU(2) \simeq S_5$
embedding	$\hookrightarrow \mathbb{R}^3$ with $x^2+y^2+z^2=r^2$	$\hookrightarrow \mathbb{C}^3$ with $z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 = \rho^2$
labelling irrep	$(l) \in \mathbb{N}_0$	$(p,q) \in \mathbb{N}_0^2$
dim(irrep)	$(2l+1)$	$(p+1)(q+1)(p+q+2)/2$
labelling states irrep	$m = -l..l$	$r = 0..q\ ,\ s = 0..p\ ,\ t = 0..(p+r-s)$

Fun example $(1,1)_{SU(3)} \longrightarrow S_3$

- Branching rule $(1,1)_{SU(3)} \longrightarrow (\mathbf{1} + \mathbf{1}' + 3\ \mathbf{2})_{S3}$ ⇒ one S_3-invariant in (1,1) basis

- This smells of eightfold way let's guess the invariant
  S_3 is a discrete flavour symmetry exchanging the x,y,z or u,d,s flavours

$$
\begin{aligned}
\mathcal{I}[S_3]_{1,1} &= xy^* + yx^* + z^*y + y^*z + x^*y + xz^* = v[S_3]_{1,1} \cdot \mathcal{B}_{(1,1)} \ , \\
v[S_3]_{1,1} &= (1,-1,0,1,0,-1,-1,-1) \ , \\
\mathcal{B}_{(1,1)} &= \left\{ xz^*, -yz^*, \frac{xx^* + yy^* - 2zz^*}{\sqrt{6}}, xy^*, \frac{xx^* - yy^*}{\sqrt{2}}, -x^*y, -y^*z, -x^*z \right\}
\end{aligned}
$$

- Construct Gell-Mann basis basis above (well in this case it's just the structure constants
  and check that S_3-generators lead to a representation of S_3
  (this representation is reducible $(1,0)_{SU(3)} \longrightarrow (\mathbf{1}+\mathbf{2})_{S3}$

VII. The tensor-generating function

- The Molien function counts the number of invariants
  Is there an object that counts the number of covariants (=tensors)?

- The (tensor)-generating function

$$M_H(\mathbf{c}, \mathbf{f}; P) = \frac{1}{|\mathcal{R}_f(h))|} \sum_{h \in H} \frac{\chi_c[h]^*}{\det(\mathbf{1} - P\mathcal{R}_f(h))} = \sum_{n \geq 0} c_n P^n$$

Thm: positive c_n number of **c**-tensor in the irrep **f** where $\chi_c[h]=tr[R_c(h)]$ is the character
N.B. reduces to Molien fct for **c=1**, **f=3**; since $\chi_1[h] = 1$

- Similar program of syzygies, primary and secondary covariants etc applies (details paper....)

Important application: branching rules

- Invariant generating fct = Molien fct ⇒ $(p,q) \rightarrow (n^{I}_{(p,q)}\mathbf{I} + \quad)_{F3}$ number of invariants in branching rule

- ⇒ **c**-Tensor generating fct ⇒ $(p,q) \rightarrow (n^{c}_{(p,q)}\mathbf{c} + \quad)_{F3}$ number of **c**-tensors in branching rule

$\boxed{\text{get the branching rules!!!}}$

- computed all tensor-generating fcts for database -- example how it works:

```
In[1]:= SetDirectory["...(your directory).../SUtree_v1p0/"];

In[2]:= $RecursionLimit=260;
        <<SUtree.m

In[3]:= BranchingSU3[{3,0}, "A4"];
Out[3]= {{3, 0}, 10, {1, 1}, {3, 1}, {3, 1}, {3, 1}}
```

Epilogue

open ends

☆ U(3) rather than SU(3)
- classification not done (Ludl'10 some progress)
- thought U(3) = U(1)xSU(3) more complicated than $F_1 x F_3$ (there can be twists)

☆ Generalization to SU(n) ... how much is known
- Hanney & He '99 "A Monograph on the classification of the discrete subgroups of SU(4)"
- Bet on "quadrihedral" groups $Z_n \, x Z_n x Z_n \rtimes Z_4$ with invariants $x^n + y^n + z^n + w^n$ and alike

☆ Language between explicit and spontaneous breaking
- how does potential look like which breaks SU(3) → F_3? *"What's the landscape?"*
- suppose explicit breaking terms are non-renormalizable
  can potential in SSB-picture be renormalizable?
  (examples in the literature give no answer ..)

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