The gradient flow and the determination of α_s

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NIC, DESY

[M. Lüscher. arXiv:1006.4518]
[M. Lüscher, P. Weisz. arXiv:1101.0963]

[Patrick Fritzsch, Alberto Ramos. arXiv:1301.4388]
[Alberto Ramos. arXiv:1308.4558]

Special Thanks to: Rainer Sommer.

Motivation

Computing the strength of fundamental interactions

- Take some experimental observable $O(\mu; p)$.
- Work hard to get

$$O(\mu; p) = A(p)\alpha_{MS}(\mu) + B(p)\alpha^2_{MS}(\mu) + \dots$$

- Determine $\alpha_{MS}(\mu)$ by comparing experiment and theory computation

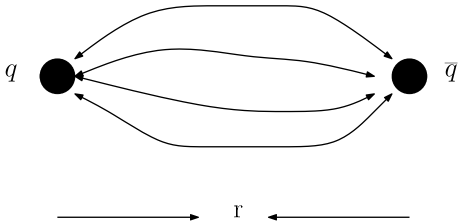
$$\begin{aligned} g_e - 2 : \alpha_{em} &= 7.297\,352\,5698(24) \times 10^{-3} & \tau : \alpha_s(M_Z) &= 0.1198(15) \\ \text{recoil} : \alpha_{em} &= 7.297\,352\,585(48) \times 10^{-3} & e^+e^- : \alpha_s(M_Z) &= 0.1172(37) \end{aligned}$$

- Caveats:
  - What about higher orders in PT?.
  - What about non-perturbative contributions?

The lattice alternative: Non-perturbative running coupling

1. Non-perturbative (physical) coupling definition: $\alpha_O(\mu) = O(\mu; p)/A(p)$
2. Finite size scaling [Lüscher, Weisz, Wolff.1991].
3. Schrödinger Functional (SF) [Lüscher, Narayanan, Weisz, Wolff. arXiv:9207009].
4. Gradient flow [M. Lüscher. arXiv:1101.0963].

The strength of YM



- Take $O(\mu) = \frac{3r^2}{4} F(r)\Big|_{\mu=1/r}$
- This defines the "potential scheme". NP coupling definition.

$$\alpha_{qq}(\mu) = \frac{3r^2}{4} F(r)\Big|_{\mu=1/r}$$

- Perturbation theory tells

$$\alpha_{qq}(\mu) = \alpha_{MS}(\mu) + c_1 \alpha_{MS}^2(\mu) + \ldots$$

Physical couplings

"Any" observable can be used for a non perturbative definition of the strong coupling, but...

... We need to evaluate $O(\mu)$ in the lattice: precision, easy to evaluate, ...

Gradient flow: basics

- Add "extra" (flow) time coordinate t ($\neq x_0$). Define gauge field $B_\mu(x,t)$

$$\begin{aligned} \frac{dB_\mu(x,t)}{dt} &= D_\nu G_{\nu\mu}(x,t) \\ G_{\nu\mu}(x,t) &= \partial_\nu B_\mu(x,t) - \partial_\nu B_\mu(x,t) + [B_\nu(x,t), B_\mu(x,t)], \quad D_\mu = \partial_\mu + [B_\mu, ...] \end{aligned}$$

  with initial condition $B_\mu(x, t=0) = A_\mu(x)$. (Lagenvin without noise!)

- Since

$$\frac{dB_\mu(x,t)}{dt} = D_\nu G_{\nu\mu}(x,t) \quad \left(\sim -\frac{\delta S_{\rm YM}[B]}{\delta B_\mu} \right)$$

  The flow field tends to a smooth classical solution of the system. UV fluctuations are more and more suppressed (more precise statement later) as t increases.

- Correlation functions of the "smooth" field $B_\mu(x,t)$

$$G(x_1, x_2, \ldots) = \langle B(x_1,t) B(x_2,t) \cdots \rangle$$

  are finite after the usual bare parameter renormalization [Lüscher, Weisz. arXiv:1101.0963].

- For example, in pure YM

$$\langle E(x,t) \rangle = \frac{1}{4} \langle G^a_{\mu\nu}(x,t) G^a_{\mu\nu}(x,t) \rangle$$

  is finite (for $t > 0$) after the usual coupling renormalization.

Gradient Flow: Some intuition

In the gradient flow equation

$$\frac{dB_\mu(x,t)}{dt} = D_\nu G_{\nu\mu}(x,t); \quad B_\mu(x,0) = A_\mu(x)$$

Expand the flow field in powers of g_0.

$$B_\mu(x,t) = \sum_{n=1}^{\infty} B_{\mu,n}(x,t) g_0^n$$

To leading order, $GF \equiv$ Heat equation ($+$ gauge terms)

$$\frac{dB_{\mu,1}(x,t)}{dt} = \partial_\nu^2 B_{\mu,1}(x,t)$$

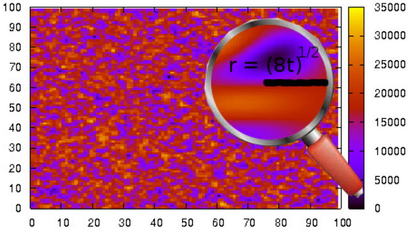
that has solution

$$B_{\mu,1}(x,t) = \int \frac{d^4p}{16\pi^2} e^{-p^2 t} e^{\imath px} \tilde{A}_\mu(p).$$

In space representation

$$B_{\mu,1}(x,t) = \frac{1}{(4\pi t)^2} \int d^4y \, e^{-\frac{(x-y)^2}{4t}} A_\mu(y)$$

We are "looking" at world with a resolution $\sim \sqrt{8t}$.



Gradient flow: Perturbative analysis

$$\frac{dB_\mu(x,t)}{dt} = D_\nu G_{\nu\mu}(x,t) + \alpha D_\mu \partial_\nu B_\nu(x,t) = \partial_\nu \partial_\nu B_\mu + (\alpha - 1)\partial_\mu \partial_\nu B_\nu + R_\mu$$

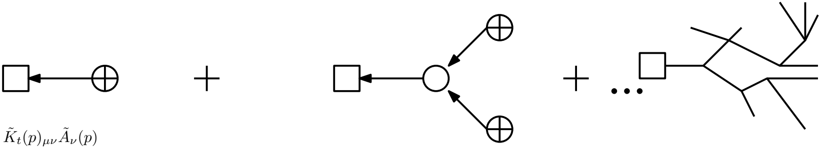
Has as solution

$$B_\mu(x,t) = \int d^4 y \left\{ K_t(x-y)_{\mu\nu} A_\nu(y) + \int_0^t ds\, K_{t-s}(x-y)_{\mu\nu} R_\nu(y,s) \right\}$$

with the heat kernel

$$K_t(z)_{\mu\nu} = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{\imath pz}}{p^2} \left\{ e^{-tp^2}(\delta_{\mu\nu} p^2 - p_\mu p_\nu) + p_\mu p_\nu e^{-\alpha t p^2} \right\}$$

We can write $\tilde{B}^a_\mu(p)$ as

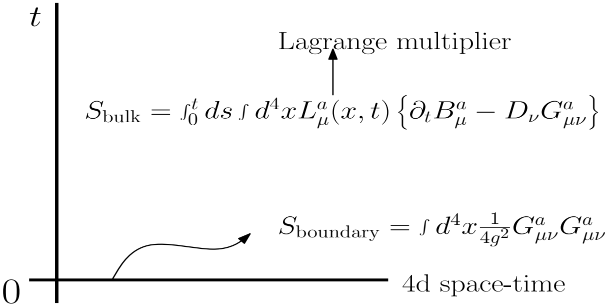


$\tilde{K}_t(p)_{\mu\nu}\tilde{A}_\nu(p)$

$\frac{1}{2}\int_0^t ds\, \tilde{K}_{t-s}(p)_{\mu\nu} \int d^4q_1 d^4q_2 \delta(p - q_1 - q_2) X^{(2,0)}(p, q_1, q_2)^{abc}_{\nu\nu_1\nu_2} \tilde{K}_s(q_1)_{\nu\nu_1} \tilde{K}_s(q_2)_{\nu\nu_2} \tilde{A}^b_{\nu_1}(q_1)\tilde{A}^c_{\nu_2}(q_2)$

Gradient Flow: Proof of finiteness

We can see the theory as a 5d local field theory [Zinn-Justin '86, Zinn-Justin, Zwanziger '88]



$$S_{\rm Total} = S_{\rm bulk} + S_{\rm boundary}$$

Theory finite to all orders of perturbation theory

- Power counting
- Theory has BRS invariance
- No extra counterterms $\Rightarrow$ Theory is finite after usual renormalization.

Gradient flow: coupling

Take the Energy density as a candidate observable

$$\langle E(t)\rangle = \frac{1}{4\mathcal{Z}}\int \mathcal{D}A_\mu\, G^a_{\mu\nu}(x,t)G^a_{\mu\nu}(x,t)e^{-S[A]}$$

In perturbation theory we have:

$$\langle E(t)\rangle = \frac{3g^2_{\overline{MS}}}{16\pi^2 t^2}(1 + c_1 g^2_{\overline{MS}} + \mathcal{O}(g^4_{\overline{MS}}))$$

and in terms of the running coupling $\alpha(\mu)$ at scale $\mu = 1/\sqrt{8t}$.

$$t^2\langle E(t)\rangle = \frac{3}{4\pi}\alpha_{\overline{MS}}(\mu)\left[1 + c_1'\alpha_{\overline{MS}}(\mu) + \mathcal{O}(\alpha^2_{\overline{MS}})\right]$$

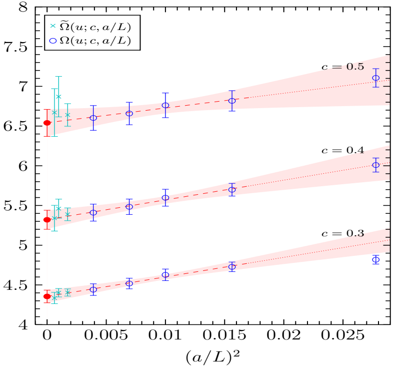
Therefore one can define the strong coupling at a scale $\mu = 1/\sqrt{8t}$

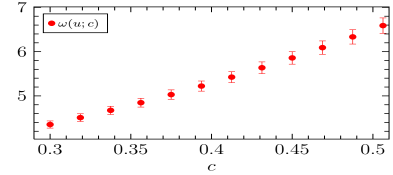
$$\alpha(\mu) = \frac{4\pi}{3}t^2\langle E(t)\rangle$$

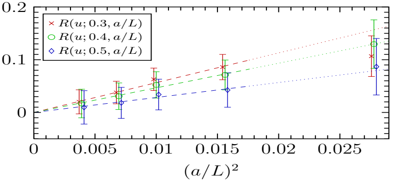
- Non-perturbative definition.
- Easy to evaluate on the lattice.
- precise (smooth observable).

Why is a good choice? $N_f = 2$ and $SU(3)$ simulations

L/a	6	8	10	12	16
β	5.2638	5.4689	5.6190	5.7580	5.9631
$\kappa_{\rm sea}$	0.135985	0.136700	0.136785	0.136623	0.136422
$N_{\rm meas}$	12160	8320	8192	8280	8460
$\bar{g}^2_{\rm SF}(L_1)$	4.423(75)	4.473(83)	4.49(10)	4.501(91)	4.40(10)
$\bar{g}^2_{\rm GF}(\mu)\ (c = 0.3)$	4.8178(46)	4.7278(46)	4.6269(47)	4.5176(47)	4.4410(53)
$\bar{g}^2_{\rm GF}(\mu)\ (c = 0.4)$	6.0090(86)	5.6985(86)	5.5976(97)	5.4837(97)	5.410(12)
$\bar{g}^2_{\rm GF}(\mu)\ (c = 0.5)$	7.106(14)	6.817(15)	6.761(19)	6.658(19)	6.602(24)

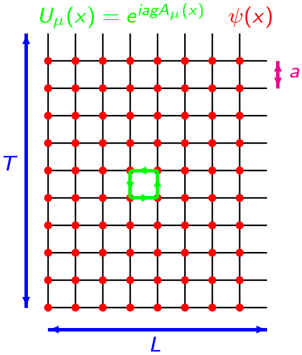






Lattice YM in one slide

Lattice field theory $\longrightarrow$ Non Perturbative definition of QFT.



$$\begin{aligned}\langle O\rangle &= \frac{1}{\mathcal{Z}}\int \mathcal{D}[U]\mathcal{D}\overline{\psi}\mathcal{D}\psi\, O(U,\overline{\psi},\psi)e^{-S_G[U]-S_F[U,\overline{\psi},\psi]} \\ &= \frac{1}{\mathcal{Z}}\int \mathcal{D}[U]\, O(U)_{\rm Wick}\, e^{-S_G[U]}\,\det(D)\end{aligned}$$

- Compute the integral numerically $\rightarrow$ Monte Carlo sampling of $e^{-S_G[U]}\det(D) \geq 0$.
- Observable computed averaging over samples

$$\langle O\rangle = \frac{1}{N_{\rm conf}}\sum_{i=1}^{N_{\rm conf}} O(U_i) + \mathcal{O}(1/\sqrt{N_{\rm conf}})$$

$$S_G[U] = \frac{\beta}{2N}\sum_{p\in\text{Plaquettes}} Tr(1 - U_p - U_p^+)$$

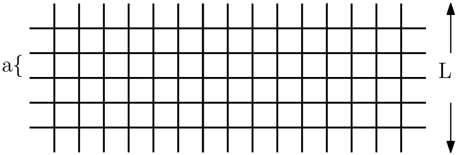
One to one relation between a and β.

Finite size scaling and step scaling function

$$\alpha(\mu) = \frac{4\pi}{3} t^2 \langle E(t) \rangle \Big|_{\sqrt{8t}=1/\mu}$$

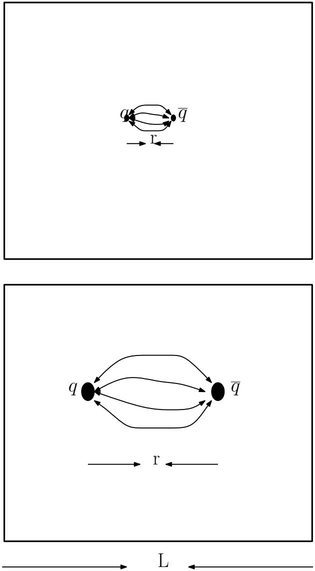
Huge computer resources

$L/a \sim 100 - 1000.$



$$a \ll \frac{1}{\mu_{\max}} < \frac{1}{\mu_{\min}} \ll L$$

Finite size scaling and step scaling function



Conditions:

- Small cutoff effects:

$$r/a \gg 1 \quad (\sim 10)$$

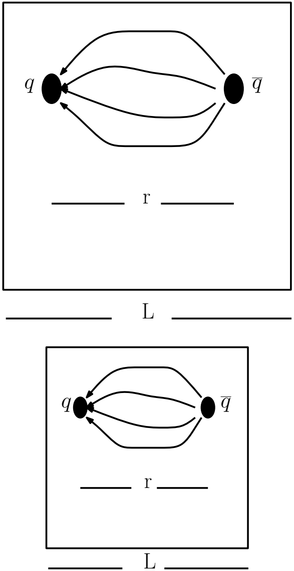
- I want to change μ from perturbative to non-perturbative: Change r by a factor 10.
- FV effects small:

$$L/a \gg r/a \quad (\sim 10)$$

Huge lattices

$L/a > 1000$

Finite size scaling and step scaling function

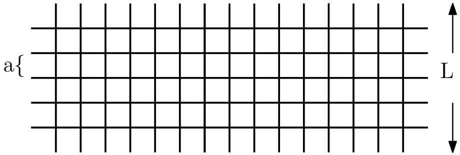


Finite volume effects part of the scheme:

- Fix
$$\mu L = \text{constant}$$
- No FV corrections.
- Only condition
$$L/a \gg 1 \quad (\sim 10)$$
- Coupling only depends on one scale: L
$$g^2(\mu)\,\text{notation} : g^2(L)$$
- Step scaling function: How much change the coupling when we change the renormalization scale:
$$\sigma(u) = g^2(2\mu)\Big|_{g^2(\mu)=u}$$
achieved by simple changing L/a!
- Recursive procedure allows study orders of magnitude change in renormalization scale *without* using large lattices.

Finite size scaling and step scaling function

$$\alpha(\mu) = \frac{4\pi}{3} t^2 \langle E(t) \rangle \Big|_{\sqrt{8t}=1/\mu}$$



$$a \ll \frac{1}{\mu_{\max}} < \frac{1}{\mu_{\min}} \ll L$$

Huge computer resources

$L/a \sim 100 - 1000$.

Finite volume renormalization schemes

- Identify finite volume with the renormalization scale $\mu = 1/\sqrt{8t} = 1/cL$.
- Coupling $\alpha(\mu)$ depends on no other scale but L (Notation: $\alpha(L)$).
- Finite Volume effects part of the scheme [Lüscher, Weisz, Wolff. 1991].
- $a \ll 1/\mu = \sqrt{8t}$ easily achieved: $L/a \sim 10 - 40$
- Boundary conditions
  - Periodic b.c: Coupling non analytic in g^2, non universal $\beta-$function, difficult P.T. [Fodor et al. [arXiv1208.1051:]]
  - Avoided with Schrödinger functional (SF) [P. Fritzsch, A.Ramos [arXiv:1301.4388]]....
  - ...and Twisted b.c. [A.Ramos [arXiv:1308.4558]]
- One has to re-do the computation

$$\alpha(L) = \mathcal{N}\, t^2 \langle E(t) \rangle \Big|_{\sqrt{8t}=cL}$$

Boundary conditions

With periodic b.c., dynamics is dominated by fluctuations constant in space

- Leading order contribution of zero momentum modes is not quadratic [A. Gonzalez-Arroyo et al. 1983]

$$S \sim [\tilde{A}_\mu(0), \tilde{A}_\nu(0)]^2.$$

  With periodic b.c. these **are not** gauge degrees of freedom.
- One has to solve these integrals to define a "propagator"

$$\langle A_\mu A_\nu \rangle \sim \int \mathcal{D}A \; A_\mu A_\nu e^{-A_\mu M A_\nu - [\tilde{A}_\mu(0), \tilde{A}_\nu(0)]^2}$$

- Convergence properties of these integrals depend on d and $SU(N)$.
- Difficult (but, in fact has been solved! [Z. Fodor et al. 2012. arXiv:1208.1051]).
- Coupling definition α_{PT} non analytic in α_{MS}.

Moral: periodic b.c. in small volume leads to

- Non analytic coupling:
  - $SU(N)$ and $N > 2$: $\alpha = \alpha_{\mathrm{MS}}(1 + \sqrt{\alpha_{\mathrm{MS}}} + \dots)$
  - $SU(2)$: $\alpha = \alpha_{\mathrm{MS}}(1 + \log \alpha_{\mathrm{MS}} \dots)$

Boundary conditions

Make constant gauge field configurations incompatible with boundary conditions.

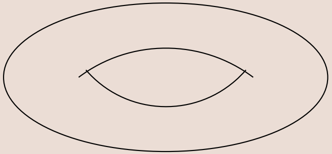
Schrödinger Functional

- SF: $L^3 \times T$ box. Dirichlet b.c. in time

$$\begin{aligned} A_i(\mathbf{x}, x_0 = 0) &= C_i(\mathbf{x}) \\ A_i(\mathbf{x}, x_0 = T) &= C_i'(\mathbf{x}) \end{aligned}$$

- If C_i, C_i' chosen wisely $\Rightarrow$ Unique gauge configuration with minimum action.

Twisted boundary conditions



- Gauge field periodic modulo g.t.

$$\begin{aligned} A_\mu(x + L\hat{\nu}) &= \Omega_\nu(x) A_\mu(x) \Omega_\nu^+(x) \\ &+ \imath \Omega_\nu(x) \partial_\mu \Omega_\nu^+(x). \end{aligned}$$

- If Ω_μ chosen wisely $\Rightarrow$ Unique gauge configuration with minimum action.

Coupling with twisted bc.

Task in life

Compute in perturbation theory

$$t^2\langle E(t)\rangle = \frac{1}{4}\langle G_{\mu\nu}(t)G_{\mu\nu}(t)\rangle = \mathcal{N}(t)\alpha_0 + \mathcal{O}(\alpha_0^2)$$

1. Compute $B_\mu(x,t)$ to leading order.
2. Compute The observable.

Coupling with twisted bc.

Key idea

In a periodic world only gauge invariant quantities need to be periodic.

$$A_\mu(x+L\hat{\nu}) = \Omega_\nu(x)A_\mu(x)\Omega_\nu^+(x) + \Omega_\nu(x)\partial_\mu\Omega_\nu^+(x) = A_\mu^{[\Omega_\nu(x)]}(x).$$

Consistency requires

$$\begin{aligned} A_\mu(x+L\hat{\nu}+L\hat{\rho}) &= A_\mu^{[\Omega_\nu(x+L\hat{\rho})\Omega_\rho(x)]}(x) = A_\mu^{[\Omega_\rho(x+L\hat{\nu})\Omega_\nu(x)]}(x) \\ \Omega_\rho(x+L\hat{\nu})\Omega_\nu(x) &= e^{2\pi\imath n_{\rho\nu}/N}\Omega_\nu(x+L\hat{\rho})\Omega_\rho(x) \end{aligned}$$

- $\Omega_\mu(x)$ are *twist matrices*. They change under gauge transformation.
- $n_{\mu\nu}$ is the *twist tensor*. Invariant under gauge transformations. Encodes physics of the twist.

Our particular setup (similar to "TPL" scheme)

- Use constant twist matrices $\Omega_\mu(x) = \Omega_\mu$.

$$A_\mu(x+L\hat{\nu}) = \Omega_\nu A_\mu(x)\Omega_\nu^+$$

  and $A_\mu(x) = 0$ compatible with bc.
- We choose to twist only the plane $x_1 - x_2$ and $n_{12} = 1$.

$$\Omega_{3,4}(x) = 1; \quad \Omega_1\Omega_2 = e^{2\pi\imath m/N}\Omega_2\Omega_1$$

Coupling with twisted bc.

Notation: $i, k = 1, 2$ and $\mu, \nu = 1, 2, 3, 4$.

$$A_\mu(x + L\hat{k}) = \Omega_k A_\mu(x) \Omega_k^+ \,,$$

Define N^2 matrices ($\tilde{p}_i = \frac{2\pi \tilde{n}_i}{NL}$ with $n_i = 0, \ldots, N-1$).

$$\Gamma(\tilde{p}) = e^{\imath \alpha(\tilde{p})} \Omega_1^{-\tilde{n}_2} \Omega_2^{\tilde{n}_1}$$

They are traceless (except $\tilde{p} = 0$), linearly independent and

$$\Omega_i \Gamma(\tilde{p}) \Omega_i^+ = e^{iL\tilde{p}_i} \Gamma(\tilde{p}) \,.$$

Therefore any gauge connection compatible with bc. can be expanded

$$A_\mu^a(x) T^a = \sum_{\tilde{p}}{}' \hat{A}_\mu(x, \tilde{p}) e^{\imath \tilde{p} x} \hat{\Gamma}(\tilde{p}).$$

with $\hat{A}_\mu(x, \tilde{p})$ (numbers) periodic in x!. $p_\mu = \frac{2\pi n_\mu}{L} \quad (n_\mu \in \mathbb{Z})\,.$

$$A_\mu^a(x) T^a = \frac{1}{L^4} \sum_{\tilde{p}}{}' \tilde{A}_\mu(p, \tilde{p}) e^{\imath (p+\tilde{p})x} \hat{\Gamma}(\tilde{p}) = \frac{1}{L^4} \sum_{P}{}' \tilde{A}_\mu(P) e^{\imath P x} \hat{\Gamma}(P) \,.$$

"Total" momentum: $P_i = p_i + \tilde{p}_i$, $P_{3,4} = p_{3,4}$. Color dof $\Leftrightarrow$ momentum dof (Large N, reduction, ...). Only constant connection: $A_\mu(x) = 0$!

Coupling with twisted bc.

$$\begin{aligned} \dot{B}_\mu(x,t) &= D_\nu G_{\nu\mu}(x,t)\,, \qquad B_\mu(x,0) = A_\mu(x)\,, \\ G_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] \end{aligned}$$

$B_\mu(x,t)$ has an asymptotic expansion in g_0

$$B_\mu(x,t) = \sum_n B_{\mu,n}(x,t) g_0^n$$

After gauge fixing and to leading order

$$\dot{B}_{\mu,1}(x,t) = \partial_\nu^2 B_{\mu,1}(x,t) \quad (B_{\mu,1}(x,0) = A_\mu(x))$$

with solution

$$B_{\mu,1}(x,t) = \frac{1}{L^4} \sum_{p,\tilde{p}\neq 0} e^{-P^2 t} \tilde{A}_\mu(P) e^{\imath P x} \hat{\Gamma}(P).$$

And finally $\langle E(t) \rangle = \frac{1}{4} \langle G_{\mu\nu}(t) G_{\mu\nu}(t) \rangle = \mathcal{E}(t) + \mathcal{O}(g_0^4)$

$$\mathcal{E}(t) = \frac{g_0^2 (d-1)}{2L^4} \sum_{p,\tilde{p}\neq 0} e^{-P^2 t}$$

Boundary conditions

Arbitrary matter content $\Rightarrow$ Same coupling definition.

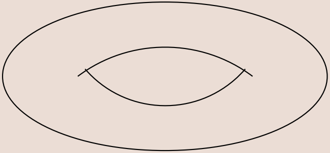
Schrödinger Functional

- Manifold with boundary: Boundary Counterterms $\mathcal{O}(a)$
- Chiral symmetry breaking $P_\pm = \frac{1}{2}(1 \pm \gamma_0)$

$$P_+\psi|_{x_0=0} = \rho_+; P_-\psi|_{x_0=T} = \rho_-$$

- Naively larger discretization effects!
- Work hard for $\mathcal{O}(a)$ improvement (PT + numerics): $c_{\rm sw}, c_t, c_{\tilde{t}}$.

Twisted boundary conditions



- Fermions in fundamental rep.

$$\psi(x + L\hat{\mu}) = \Omega_\mu \psi(x)$$

And therefore

$$\begin{aligned} \psi(x + L\hat{1} + L\hat{2}) &= \Omega_1\Omega_2\psi(x) \\ \psi(x + L\hat{2} + L\hat{1}) &= \Omega_2\Omega_1\psi(x) \end{aligned}$$

- Only way out: $SU(N)$ with N_f and $N_f/N \in \mathbb{Z}$ [Parisi '83].

Lattice computations

Lattice: Different discretization for simulation, flow and observable:

$$
\begin{aligned}
S_a[\tilde{A}_\mu] &= \frac{1}{2}\sum_P{}' \tilde{A}_\mu(P)K^{(a)}_{\mu\nu}(P)\tilde{A}_\nu(P) + \mathcal{O}(g_0^2) \\
S_f[\tilde{A}_\mu] &= \frac{1}{2}\sum_P{}' \tilde{A}_\mu(P)K^{(f)}_{\mu\nu}(P)\tilde{A}_\nu(P) + \mathcal{O}(g_0^2) \\
S_O[\tilde{A}_\mu] &= \frac{1}{2}\sum_P{}' \tilde{A}_\mu(P)K^{(O)}_{\mu\nu}(P)\tilde{A}_\nu(P) + \mathcal{O}(g_0^2)\,.
\end{aligned}
$$

Inverse of $K^{(a,f,O)}_{\mu\nu}$ are $D^{(a,f,O)}_{\mu\nu}$. Flow field:

$$
\tilde{B}_{\mu,1}(P) = \left(\exp\{tK^{(f)}(P)\}\right)_{\mu\nu}\tilde{A}_\nu(P)\,,
$$

and energy density

$$
\mathcal{E}(t,a/L) = \frac{g_0^2}{L^3}\sum_P{}' \mathrm{Tr}\left[K^{(O)}(P)\exp\{tK^{(f)}(P)\}D^{(a)}(P)\exp\{K^{(f)}_{\nu\beta}(P)\}\right] \qquad (1)
$$

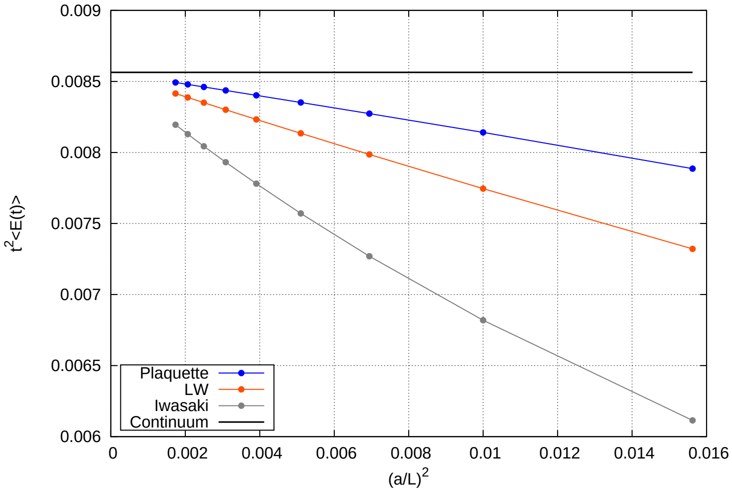
Example: Wilson, Wilson, Clover
$\left(\hat{P}_\mu = \frac{2}{a}\sin\left(a\frac{P_\mu}{2}\right); \mathring{P}_\mu = \frac{1}{a}\sin\left(aP_\mu\right); C_\mu = \cos\left(a\frac{P_\mu}{2}\right)\right)$

$$
\hat{\mathcal{E}}_{\rm clover}(t,a/L) = \frac{g_0^2}{2L^4}\sum_{p,\tilde{p}\neq 0} e^{-\hat{P}^2 t}\frac{\mathring{P}^2C^2 - \sum_\mu(\mathring{P}_\mu C_\mu)^2}{\hat{P}^2},
$$

Risks of partial improvement (SF, $c = 0.3$)

Risk of partial Improvement. Comparison of

- Wilson, Wilson, Clover.
- Lüscher-Weisz (TL improved), Wilson, Clover.
- Iwasaki, Wilson, Clover.



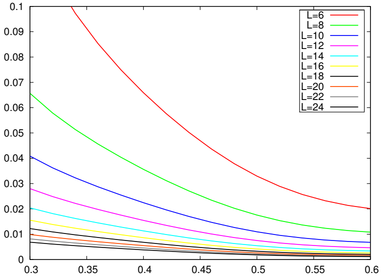
Coupling definition

Reading the value of $\langle E(t)\rangle$ at $\sqrt{8t} = cL$, we define

$$\mathcal{N}_T(c) = \frac{(d-1)c^4}{128}\sum_P{}' e^{-\frac{c^2L^2}{4}P^2} = \frac{g_0^2(d-1)c^4}{128}\sum_{n_\mu=-\infty}^{\infty}\sum_{\tilde{n}_i=0}^{N-1'} e^{-\pi^2c^2(n^2+\tilde{n}^2/N^2+2\tilde{n}_in_i/N)}$$

Twisted gradient flow coupling

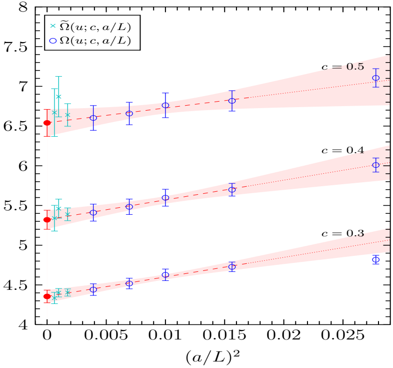
$$g^2_{TGF}(L) = \mathcal{N}_T^{-1}(c)t^2\langle E(t)\rangle\Big|_{\sqrt{8t}=cL} = g^2_{\overline{\rm MS}} + \mathcal{O}(g^4_{\overline{\rm MS}})$$

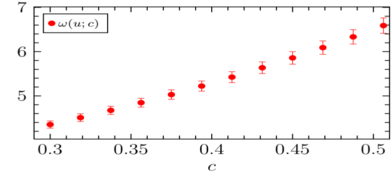


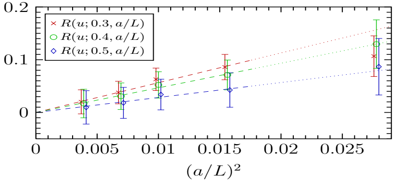
- Different c's: different schemes.
- Larger c smaller cutoff effects (more smooth).
- Larger c larger autocorrelations.
- Larger c smaller signal to noise.
- $c \in [0.3, 0.5]$ reasonable range.

SF $N_f = 2$ and $SU(3)$ simulations: $L =$ constant

L/a	6	8	10	12	16
β	5.2638	5.4689	5.6190	5.7580	5.9631
$\kappa_{\rm sea}$	0.135985	0.136700	0.136785	0.136623	0.136422
$N_{\rm meas}$	12160	8320	8192	8280	8460
$\bar{g}^2_{\rm SF}(L_1)$	4.423(75)	4.473(83)	4.49(10)	4.501(91)	4.40(10)
$\bar{g}^2_{\rm GF}(\mu)\ (c = 0.3)$	4.8178(46)	4.7278(46)	4.6269(47)	4.5176(47)	4.4410(53)
$\bar{g}^2_{\rm GF}(\mu)\ (c = 0.4)$	6.0090(86)	5.6985(86)	5.5976(97)	5.4837(97)	5.410(12)
$\bar{g}^2_{\rm GF}(\mu)\ (c = 0.5)$	7.106(14)	6.817(15)	6.761(19)	6.658(19)	6.602(24)

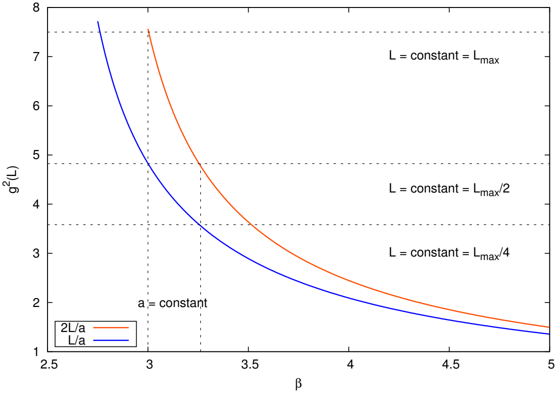






Running coupling

$$\beta \Longleftrightarrow a; \qquad g^2(L) \Longleftrightarrow L \Longleftrightarrow \mu$$



Step scaling function

$$\sigma^{-1}(u,a) = g^2(L/2)\Big|_{g^2(L)=u}$$

Continuum limit

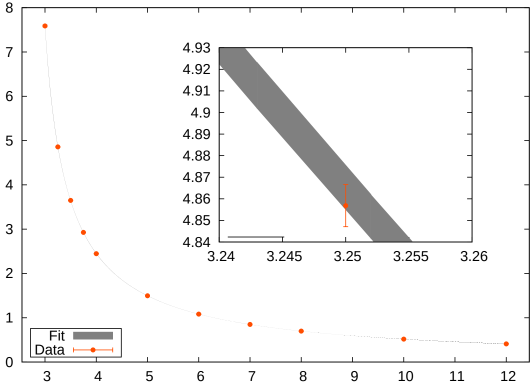
$$\sigma^{-1}(u) = \lim_{a\to 0} \sigma^{-1}(u,a)$$

Simulate several pair of lattices

$SU(2)$ YM running coupling

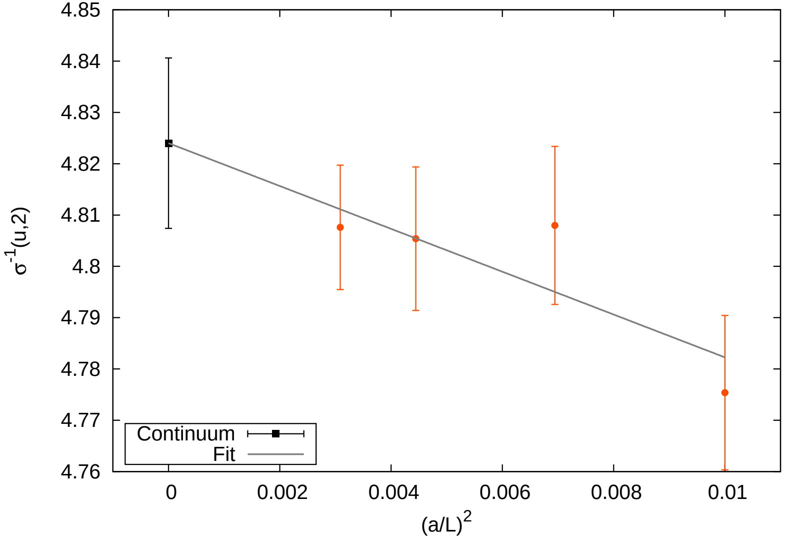
- Simulations for $L/a = 10, 12, 15, 18, 20, 24, 30, 36$ at $\beta \in [2.75, 12]$.
- Modest statistics: 2048 independent measurements of g^2_{TGF}.
- Between 0.15-0.25% precision in g^2_{TGF} *for all* L/a.
- Padè fit (constrain to PT), 4 parameters, $\chi^2/\mathrm{ndof} = 5.9/7$.
- Example: $L/a = 36$

β	$g^2_{TGF}(L)$
12.0	0.41078(64)
10.0	0.51809(83)
8.0	0.6987(11)
7.0	0.8497(13)
6.0	1.0819(18)
5.0	1.4968(25)
4.0	2.4465(44)
3.75	2.9277(54)
3.5	3.6494(69)
3.25	4.8568(99)
3.0	7.587(20)
2.9	10.610(32)
2.8	16.752(47)
2.75	22.168(59)



Step scaling function

- Modest cutoff effects. Starting recursion with $u = 7.5$.



Step scaling function

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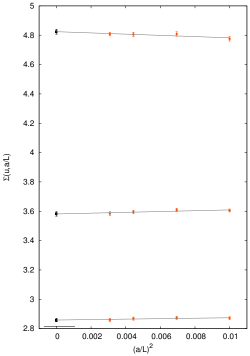
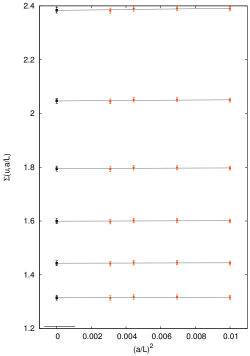
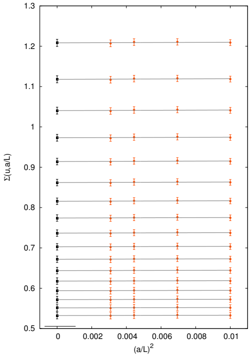
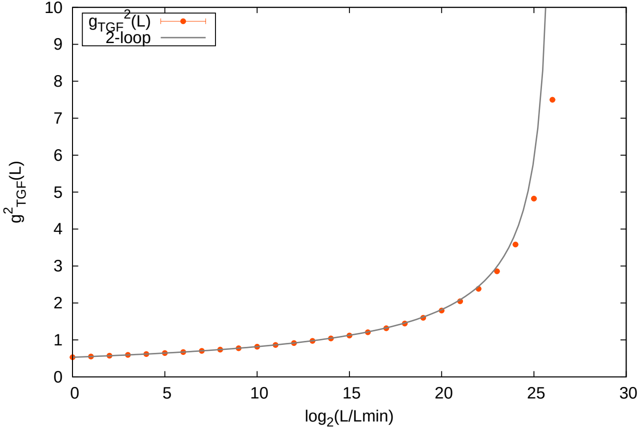






$g^2_{TGF}(L)$ for pure gauge $SU(2)$



Since $\Lambda = \mu (b_0 g^2(\mu))^{-b1/2b_0^2} e^{-1/2b_0 g^2(\mu)} e^{-\int_0^{g^2(\mu)} \left\{ \frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x} \right\}}$ and $\mu = 1/cL$.

$$\Lambda L_{\max} = 1.509(44) \quad (@g^2_{TGF}(L) = 1.7948(93))$$

Determine $M_p L_{\max} = (aM_p)(L_{\max}/a)$ and use $M_p^{\exp} = 938\,\mathrm{MeV} \Rightarrow L_{\max}$ in MeV.

Conclusions

Gradient flow

New tool to study non-perturbative aspects of strongly coupled gauge theories:

- Renormalization is simplified.
- Composite operators do not need renormalization.
- Flow fermions [M. Lüscher. arXiv:302.5246]: No operator mixing

$$(\partial_t - D_\mu D_\mu)\psi = 0$$

- Solid theoretical understanding.
- Applications outside lattice? Effective field theories? Large N? Reduction?
- For any renormalization problem: Can I solve at positive t?

Gradient flow running couplings

- Theoretically appealing method to compute strength of interactions.
- From the perturbative to the non-perturbative regimes.
- May lead to a precise determination of α_s.
- Applications BSM: Conformal field theories, walking technicolor, etc...