From multiple unitarity cuts to the coproduct of scalar Feynman integrals

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IPhT SACLAY

Feynman diagrams and amplitudes

- Diagrammatic representation of the perturbative expansion of a field theory.
- Amplitude: sum of Feynman diagrams contributing to a $n \rightarrow m$ process.
- Feynman diagrams can be classified according to the **number of loops** and the **number of external legs**. Their calculation greatly increases with both but is necessary for precise prediction for experiments.
- Two complimentary ways to optimise their calculation:
    - ➡ **Mathematics**: a large class of Feynman diagrams can be written in terms of a specific class of functions.
    - ➡ **Physics**: Feynman diagrams describe physical processes.

Outline

1. Multiple Polylogarithms (MPLs).
2. The Hopf algebra, coproduct (and symbol) of MPLs.
3. Single unitarity cuts and the largest time equation.
4. Multiple unitarity cuts.
5. Applications.
6. Conclusions and outlook.

Multiple Polylogarithms

- A large class of Feynman integrals can be written in terms of multiple polylogarithms, defined recursively by the iterated integral:

$$\boxed{G(a_1,\ldots,a_n;z)=\int_0^z \frac{dt}{t-a_1}G(a_2,\ldots,a_n;t)} \qquad a_i,z\in\mathbb{C}$$

  - ex: Classical polylogarithms:

$$G\left(\vec{0}_n;z\right)=\frac{1}{n!}\log^n z \qquad G\left(\vec{a}_n;z\right)=\frac{1}{n!}\log^n\left(1-\frac{z}{a}\right) \qquad G\left(\vec{0}_{n-1},a;z\right)=-\mathrm{Li}_n\left(\frac{z}{a}\right)$$

- Transcendental weight:

$$w\left(G\left(\vec{0}_n;z\right)\right)=n \qquad w\left(\zeta_n\right)=n \qquad w\left(\pi^n\right)=n$$

A. B. Goncharov, *Multiple polylogarithms, cyclotomy and modular complexes*, Math.Res.Lett. 5 (1998) 497–516
A. B. Goncharov, *Multiple polylogarithms and mixed Tate motives*
A. B. Goncharov, M. Spradlin, C. Vergu, and A. Volovich, *Classical Polylogarithms for Amplitudes and Wilson Loops*, Phys.Rev.Lett. 105 (2010) 151605

Multiple Polylogarithms

- Iterative definitions is an integration rule: provided an integrand has been written in the correct form (!), the integration is trivial.

- This would be of limited interest for analytical calculation if we did not know how to work with the very large class of MPLs

  - ex: a 17 page result was reduced to a couple of lines by identifying using relations between polylogarithms, see

V. Del Duca, C. Duhr and V. A. Smirnov, *An Analytic Result for the Two-Loop Hexagon Wilson Loop in N = 4 SYM*, JHEP 1003 (2010) 099
V. Del Duca, C. Duhr and V. A. Smirnov, *The Two-Loop Hexagon Wilson Loop in N = 4 SYM*, JHEP 1005 (2010) 084
A. B. Goncharov, M. Spradlin, C. Vergu, and A. Volovich, *Classical Polylogarithms for Amplitudes and Wilson Loops*, Phys.Rev.Lett. 105 (2010) 151605

- It is an empirical observation that most Feynman diagrams can be written in terms of a limited subset of all MPLs (~ classical and harmonic polylogarithms)

The Hopf algebra and coproduct of MPLs

C. Duhr, H. Gangl, J.R. Rhodes, From polygons and symbols to polylogarithmic functions, JHEP 1210 (2012) 075
C. Duhr, *Hopf algebras, coproducts and symbols: an application to Higgs boson amplitudes*, JHEP 1208 (2012) 043

- The space spanned by all MPLs has a very rich structure. In particular, the space of *H* of all HPLs modulo π forms a *Hopf algebra,* which can be equipped with a *coproduct* Δ:

$$\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$$

- Properties
  - Coassociative: $$(\mathrm{id} \otimes \Delta)\,\Delta = (\Delta \otimes \mathrm{id})\,\Delta$$
  - Respects multiplication: $$\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$$
  - Respects the weight: $$\mathcal{H}_n \xrightarrow{\Delta} \bigoplus_{k=0}^{n} \mathcal{H}_k \otimes \mathcal{H}_{n-k}\,.$$

- Examples

$$\Delta(\log z) = 1 \otimes \log z + \log z \otimes 1$$

$$\Delta\left(\mathrm{Li}_n(z)\right) = 1 \otimes \mathrm{Li}_n(z) + \sum_{k=0}^{n-1} \mathrm{Li}_{n-1}(z) \otimes \frac{\log^k z}{k!}$$

The symbol and the coproduct

- The symbol corresponds to the maximum iteration of the coproduct, and coassociativity guarantees it is uniquely defined.

$$\mathcal{S}(F_n) \sim \Delta_{1,\ldots,1}(F_n)$$

- Manipulations of the symbol tensor:

$$\ldots \otimes \log(a\, b) \otimes \ldots = \ldots \otimes a \otimes \ldots + \ldots \otimes b \otimes \ldots$$

$$\ldots \otimes \rho_n \otimes \ldots = 0 \qquad\qquad (\rho_n)^n = 1$$

$$S(\pi^n F) = S(\zeta_n F) = 0$$

The symbol and the coproduct

- It is consistent to slightly modify the action of the coproduct so that, e.g.:

$$\Delta(\pi) = \pi \otimes 1 \qquad\qquad \Delta(\zeta_2) = \zeta_2 \otimes 1$$

- The coproduct keeps more information than the symbol. E.g.:

$$\mathcal{S}(\pi F) = 0 \qquad\qquad \Delta(\pi F) = \pi \otimes F$$

C. Duhr, *Hopf algebras, coproducts and symbols: an application to Higgs boson amplitudes*, JHEP 1208 (2012) 043

The integrability condition

- "Integration" of the symbol (finding a function whose symbol matches a given tensor) is a complicated problem, not solved in general. Not all tensors are the symbol of a function, they have to satisfy the integrability condition:

$$\sum_{1,\dots,n} c_{1\dots n} \, (d\log\omega_k \wedge d\log\omega_{k+1}) \log\omega_1 \otimes \cdots \otimes \log\omega_{k-1} \otimes \log\omega_{k+2} \otimes \cdots \log\omega_n = 0$$

- Example:

$$\log(x) \otimes \log(y) \qquad \text{vs} \qquad \log(x) \otimes \log(y) + \log(y) \otimes \log(x)$$

The coproduct, discontinuities and derivatives

- The two following identities are conjecture to hold:

$$\Delta \frac{\partial}{\partial z} = \left(\mathrm{id} \otimes \frac{\partial}{\partial z}\right) \Delta \qquad\qquad \Delta \mathrm{Disc} = (\mathrm{Disc} \otimes \mathrm{id})\, \Delta$$

  - Derivatives act on the last entry of the coproduct
  - Discontinuities act only on the first entry of the coproduct

- In particular, recursive discontinuities are given by:

$$\mathrm{Disc}_{x_1,\ldots,x_k} \sim \delta_{x_1,\ldots,x_k}$$

- Notation:

$\delta_{x_1,\ldots,x_k}$ is the weight (n-k) cofactor of $\Delta_{1,\ldots,1,n-k}$

Physical constraints: the first entry condition

- Feynman diagrams have branch points at the threshold for the production of new on-shell states: in massless theories, the first entry of the coproduct must be (the logarithmic of) a kinematic invariant.

- For these diagrams, the weight (1,n-1) component of the coproduct is:

$$\sum_i \log(s_i) \otimes \delta_{s_i}$$

  This is the **first entry condition**.

- This is not true for other entries of the symbol/coproduct: understanding what can appear there is an interesting problem in itself.

L.D. Landau, *Nuclear Phys.* **13** (1959)
Eden, Landshoff, Olive, Polkinghorne, *The Analytic S-Matrix, 1966*
D. Gaiotto, J. Maldacena, A. Sever, and P. Vieira, *Pulling the straps of polygons*, JHEP 1112 (2011) 011

The largest time equation

G. 't Hooft and M. Veltman, *DIAGRAMMAR*, NATO Adv.Study Inst.Ser.B Phys. 4 (1974) 177–322
M. Veltman, *Diagrammatica: The Path to Feynman rules*, Cambridge Lect.Notes Phys. 4 (1994) 1–284
E. Remiddi, *Dispersion Relations for Feynman Graphs*, Helv.Phys.Acta 54 (1982) 364

- The largest time equation is:

$$\boxed{F + F^* = -\sum_s \mathrm{Cut}_s F \quad \longrightarrow \quad \mathrm{Disc}_s F = -\mathrm{Cut}_s F}$$

- It allows to:

1. compute the discontinuity across the physical branch cuts of a Feynman diagram

2. prove the existence of a dispersive representation of individual Feynman diagrams (an not amplitudes as the optical theorem, which is implied by the largest time equation).

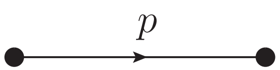
$$\boxed{F(p_i^2) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\mathrm{d}s}{s - p_1^2}\, \mathrm{Disc}_{p_1^2} F(p_i^2)}$$

W. van Neerven, *Dimensional Regularization of Mass and Infrared Singularities in Two Loop On-shell Vertex Functions*, *Nucl. Phys.* **B268** (1986)
P. Ball, V. M. Braun, H. G. Dosch, Form-factors of semileptonic D decays from QCD sum rules, Phys.Rev. **D44**, 1991

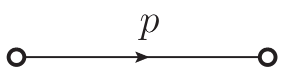
Cutting rules (scalar) — single cut: Cutkosky rules

R.E. Cutkosky, *Singularities and discontinuities of Feynman amplitudes*, *J. Math. Phys.*, **1** (1960)
L.D. Landau, *On analytic properties of vertex parts in quantum field theory*, *Nuclear Phys.* **13** (1959)

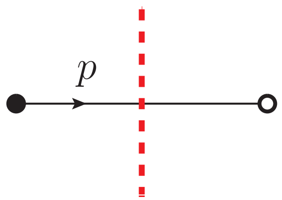
$\bullet \quad = i \qquad\qquad \circ \quad = -i$



$$= \frac{i}{p^2 + i\varepsilon}$$

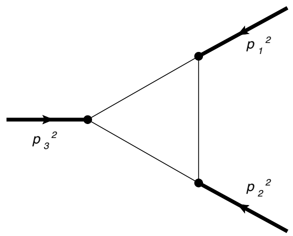


$$= \frac{-i}{p^2 - i\varepsilon}$$

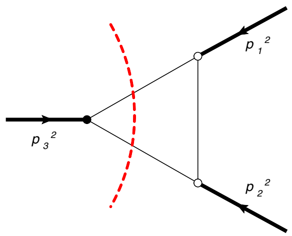


$$= 2\pi\delta(p^2)\theta(p_0)$$

Example: $\mathrm{Disc}_{p_3^2}$



$= -$



Three definitions of discontinuities

- Discontinuity across branch cuts: $\mathrm{Disc}_s F(s \pm i0) = \lim_{\varepsilon \to 0} [F(s \pm i\varepsilon) - F(s \mp i\varepsilon)]$

$$\boxed{\mathrm{Disc}_{x_1,\ldots,x_k} F = \mathrm{Disc}_{x_k} \left(\mathrm{Disc}_{x_1,\ldots,x_{k-1}} F \right)}$$

- From the coproduct:

$$\Delta_{\underbrace{1,1,\ldots,1}_{k \text{ times}},n-k} F = \sum_{\{a_1,\ldots,a_k\}} \log a_1 \otimes \cdots \otimes \log a_k \otimes g_{a_1,\ldots,a_k}$$

$$\delta(x_i, a_i) = \begin{cases} 1\,, & \text{if } a_{i|x_i=0} = 0 \\ -1\,, & \text{if } a_{i|x_i=0} = \infty \\ 0\,, & \text{otherwise} \end{cases}$$

$$\boxed{\delta_{x_1,\ldots,x_k} F \cong \sum_{\{a_1,\ldots,a_k\}} \delta(x_1, a_1) \ldots \delta(x_k, a_k)\, g_{a_1,\ldots,a_k}}$$

- Cuts:

$$\boxed{\mathrm{Cut}_{s_1,\ldots,s_k} F}$$

Three definitions of discontinuities

- Single cut:

$$\boxed{\mathrm{Disc}_s F = -2\pi i\,\theta(s)\,\delta_s F = -\mathrm{Cut}_s F}$$

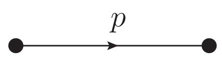
- How does this generalise for multiple discontinuities (kinematic channels no longer the simpler variables)?

- How to compute multiple cuts?

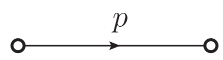
Multiple unitarity cuts

$= i$

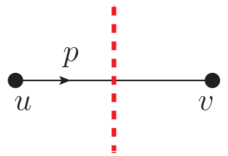
$= -i$



$$= \frac{i}{p^2 + i\varepsilon}$$



$$= \frac{-i}{p^2 - i\varepsilon}$$



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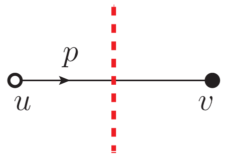

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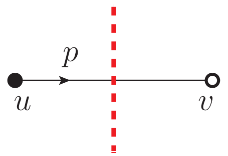

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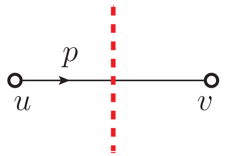
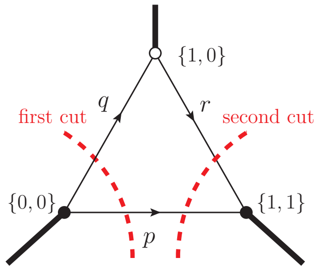


$$= 2\pi\,\delta(p^2) \prod_{i:c_i(u)\neq c_i(v)} \theta\left([c_i(v) - c_i(u)]p_0\right)$$

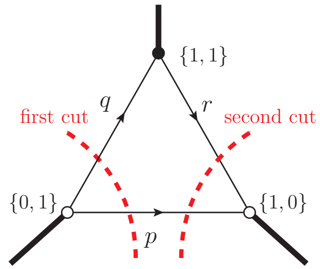
- Crossed cuts are excluded.
- Cuts in which all channels are not distinct are excluded.
- Insist on the use of real kinematics for external and loop momenta.
- These are very restrictive rules.

Multiple unitarity cuts —Examples

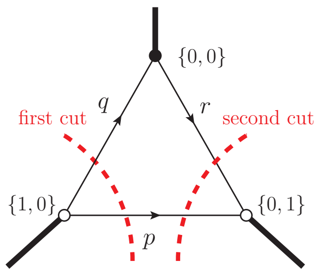
- Sequential cut of three mass triangle:



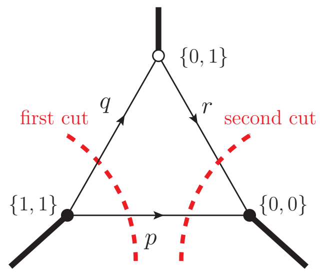
$\theta(p_0)\theta(q_0)\theta(r_0)$



$\theta(p_0)\theta(-p_0) = 0$

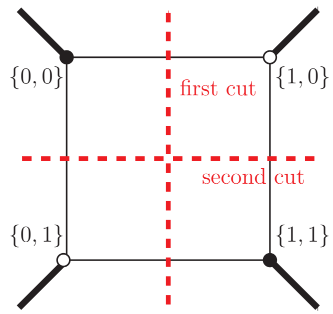


$\theta(-p_0)\theta(p_0) = 0$



$\theta(-p_0)\theta(-q_0)\theta(-r_0)$

- Vanishing crossed cut:



Relation between different discontinuities

- Relation between multiple discontinuities across branch cuts and cuts of diagrams

$$\boxed{\mathrm{Cut}_{s_1,\ldots,s_k} F = \sum_{x_1,\ldots,x_k} (-1)^k \, \mathrm{Disc}_{x_1,\ldots,x_k} F}$$

- Relation between multiple discontinuities and coproduct entries

$$\boxed{\mathrm{Disc}_{x_1,\ldots,x_k} F \cong \Theta \sum_{x_1,\ldots,x_k} \pm (2\pi i)^k \delta_{x_1,\ldots,x_k} F}$$

- The $i\varepsilon$ prescription of the x_i is inherited from that of the s_i. Determining which x_i corresponds to a given s_i must be done case by case.

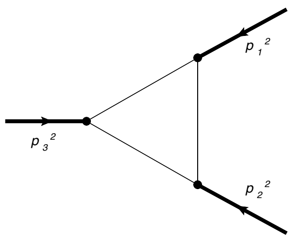
Example 1: the three-mass triangle

- Single cut of the three mass triangle:

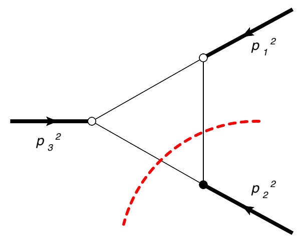
$$\mathrm{Cut}_{p_2^2} T(p_1^2, p_2^2, p_3^2) = \frac{2\pi}{\sqrt{\lambda(p_1^2, p_2^2, p_3^2)}} \log \frac{p_1^2 - p_2^2 + p_3^2 - \sqrt{\lambda(p_1^2, p_2^2, p_3^2)}}{p_1^2 - p_2^2 + p_3^2 + \sqrt{\lambda(p_1^2, p_2^2, p_3^2)}}$$

- The invariants $p^2{}_i$ are clearly not the most suitable variables; let's define

$$z, \bar{z} = \frac{p_1^2 + p_2^2 - p_3^2 \pm \sqrt{(p_1^2 + p_2^2 - p_3^2)^2 - 4p_1^2 p_2^2}}{2p_1^2}$$



$$\boxed{\mathrm{Cut}_{p_2^2} T(p_1^2, p_2^2, p_3^2) = \frac{2\pi}{p_1^2(z - \bar{z})} \ln \frac{1 - z}{1 - \bar{z}}}$$



- Cuts provide a way to identify the x_i without evaluating the full diagram.

$$\boxed{x_i \in \{z, \bar{z}, 1 - z, 1 - \bar{z}\}}$$

Example 1: the three-mass triangle

- The variables introduced for the cut triangle are indeed the most suitable for the uncut triangle.

$$T(p_1^2, p_2^2, p_3^2) = -\frac{i}{p_1^2}\frac{2}{z-\bar{z}}\mathcal{P}_2(z) \qquad \mathcal{P}_2(z) = \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\ln(z\bar{z})\ln\left(\frac{1-z}{1-\bar{z}}\right)$$

- Coproduct of the three mass triangle.

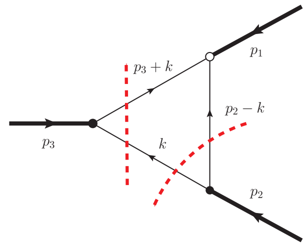
$$\begin{aligned}\Delta\left[\mathcal{P}_2(p_1^2, p_2^2, p_3^2)\right] &= \mathcal{P}_2(z)\otimes 1 + 1\otimes\mathcal{P}_2(z) + \frac{1}{2}\ln\left(-p_2^2\right)\otimes\ln\frac{1-z}{1-\bar{z}} + \frac{1}{2}\ln\left(-p_3^2\right)\otimes\ln\frac{\bar{z}}{z}\\ &+ \frac{1}{2}\ln(-p_1^2)\otimes\ln\frac{1-1/\bar{z}}{1-1/z}\end{aligned}$$

F. Chavez and C. Duhr, *Three-mass triangle integrals and single-valued polylogarithms*, JHEP 1211 (2012) 114

Example 1: the three-mass triangle

- Double cut of the three mass triangle:

$$\boxed{\mathrm{Cut}_{p_3^2,p_2^2} T = \frac{4\pi^2 i}{p_1^2 (z-\bar{z})}}$$



- Relation between cuts and discontinuities across branch cuts:

$$\boxed{\mathrm{Disc}_{p_2^2} T = -\mathrm{Cut}_{p_2^2} T \qquad \mathrm{Disc}_{p_2^2,1-z} T = (-1)^2 \mathrm{Cut}_{p_2^2,p_3^2} T}$$

- Relation between the coproduct and discontinuities across branch cuts:

$$\boxed{\mathrm{Disc}_{p_2^2} \mathcal{P}_2 \cong (-2\pi i)\, \Theta\, \delta_{p_2^2} \mathcal{P}_2 \qquad \mathrm{Disc}_{p_2^2,1-z} \mathcal{P}_2 \cong 4\pi^2\, \Theta\, \delta_{p_2^2,1-z} \mathcal{P}_2 = -2\pi^2}$$

Example 1: the three-mass triangle

- In this particular example, we can cut through all transcendental functions and get to the rational prefactor — the **leading singularity**. This is not always possible with our cutting rules.

- Single (and double) dispersion representation of the triangle particularly simple in terms of the correct variables.

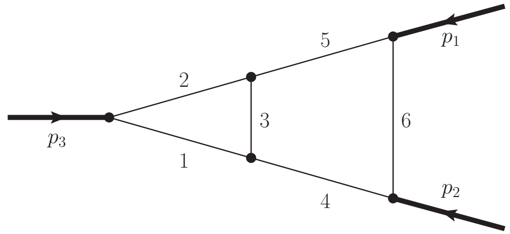
$$\boxed{T(p_1^2, p_2^2, p_3^2) = \frac{-i}{p_1^2}\frac{1}{z-\bar{z}}\int_0^1 dw \left(\frac{1}{w-\bar{z}} - \frac{1}{w-z}\right)\left[2\ln(1-w) - \ln\frac{p_3^2}{p_1^2}\right]}$$

$$\boxed{T(p_1^2, p_2^2, p_3^2) = \frac{-i}{p_1^2}\int_1^\infty dw \int_{-\infty}^0 d\bar{w}\, \frac{1}{w\bar{w}-z\bar{z}}\,\frac{1}{(1-w)(1-\bar{w})-(1-z)(1-\bar{z})}}$$

Example 2: three-mass three-point ladder

N. Usyukina and A. I. Davydychev, *An Approach to the evaluation of three and four point ladder diagrams*, Phys.Lett. B298 (1993) 363–370

- The most convenient variables are the same as for the three mass triangle.



$$\boxed{T_L(p_1^2,p_2^2,p_3^2) = i\left(p_1^2\right)^{-2}\frac{1}{(1-z)(1-\bar z)(z-\bar z)}F(z,\bar z)}$$

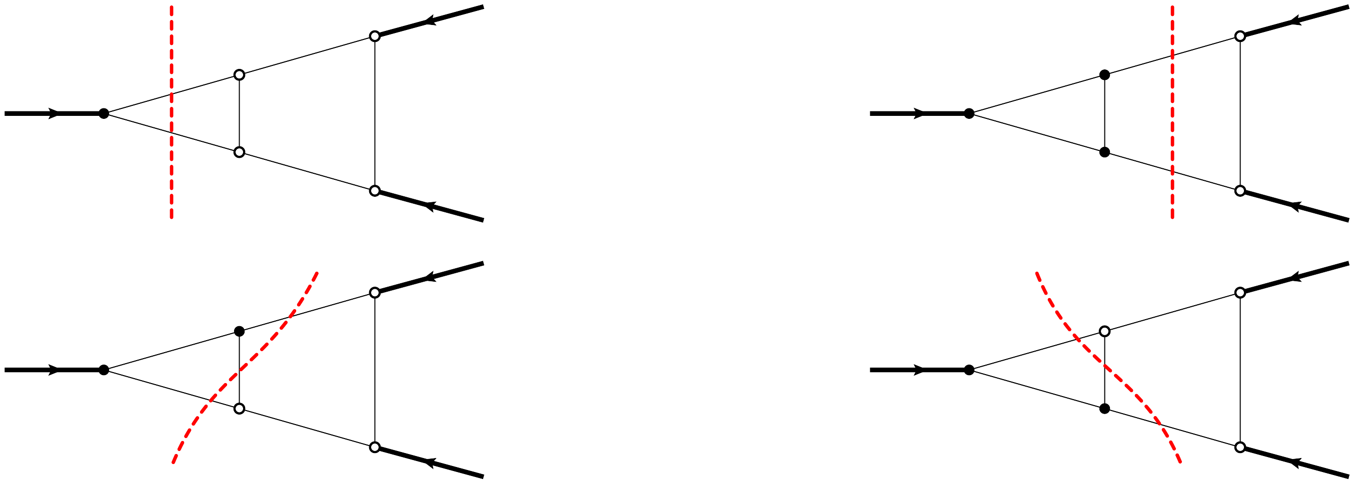
$$\begin{aligned}F(z,\bar z) =& 6\left[\mathrm{Li}_4(z)-\mathrm{Li}_4(\bar z)\right]-3\ln(z\bar z)\left[\mathrm{Li}_3(z)-\mathrm{Li}_3(\bar z)\right]\\ &+\frac{1}{2}\ln^2(z\bar z)\left[\mathrm{Li}_2(z)-\mathrm{Li}_2(\bar z)\right]\end{aligned}$$

$$\Delta_{1,3}(F(z,\bar z)) = \ln\frac{p_2^2}{p_1^2}\otimes\left[-3\,\mathrm{Li}_3(z)+3\,\mathrm{Li}_3(\bar z)+\ln\frac{p_2^2}{p_1^2}\ (\mathrm{Li}_2(z)-\mathrm{Li}_2(\bar z))\right]+\ln\frac{p_3^2}{p_1^2}\otimes\frac{1}{2}\ln z\ln\bar z\ln\frac{z}{\bar z}$$

$$\begin{aligned}\Delta_{1,1,2}(F(z,\bar z)) =& \ln\frac{p_3^2}{p_1^2}\otimes\ln z\otimes\left(\ln z\ln\bar z-\frac{1}{2}\ln^2\bar z\right)-\ln\frac{p_3^2}{p_1^2}\otimes\ln\bar z\otimes\left(\ln z\ln\bar z-\frac{1}{2}\ln^2 z\right)\\ &-\ln\frac{p_2^2}{p_1^2}\otimes\ln(1-z)\otimes\left(\ln z\ln\bar z-\frac{1}{2}\ln^2 z\right)+\ln\frac{p_2^2}{p_1^2}\otimes\ln(1-\bar z)\otimes\left(\ln z\ln\bar z-\frac{1}{2}\ln^2\bar z\right)\\ &+\ln\frac{p_2^2}{p_1^2}\otimes\ln(z\bar z)\otimes\left[\mathrm{Li}_2(z)-\mathrm{Li}_2(\bar z)\right]\end{aligned}$$

Example 2: three-mass three-point ladder

- p^2_3-channel cuts

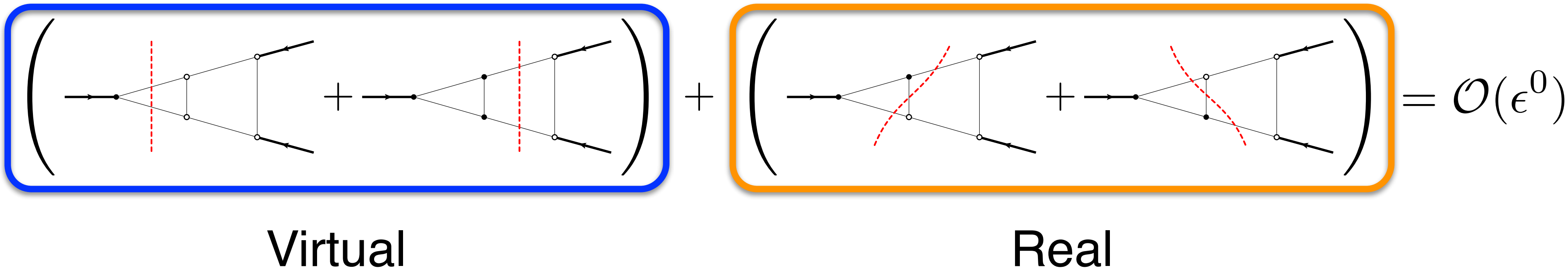


- p^2_2-channel cuts

Example 2: three-mass three-point ladder

- All the cut diagrams are divergent in d=4 dimensions (massless internal legs). The calculation is done in d=4-2**ϵ** dimensions.

- The sum of cuts on a given channel must be finite: the cancelation of divergences follows a pattern similar to the cancelation between the real and virtual contributions to a cross-section.



Example 2: three-mass three-point ladder

- $p^2{}_3$-channel cuts

$$\mathrm{Cut}_{p_3^2} T_L(p_1^2, p_2^2, p_3^2) = -\frac{\pi (p_1^2)^{-2}}{(1-z)(1-\bar{z})(z-\bar{z})} \ln z \ln \bar{z} \ln \frac{z}{\bar{z}}$$

$$\boxed{\begin{aligned} \mathrm{Cut}_{p_3^2} T_L(p_1^2, p_2^2, p_3^2) &= -\mathrm{Disc}_{p_3^2} T_L(p_1^2, p_2^2, p_3^2) \\ &\cong -2\pi \left(p_1^2\right)^{-2} \frac{1}{(1-z)(1-\bar{z})(z-\bar{z})} \delta_{p_3^2} F(z, \bar{z}) \end{aligned}}$$

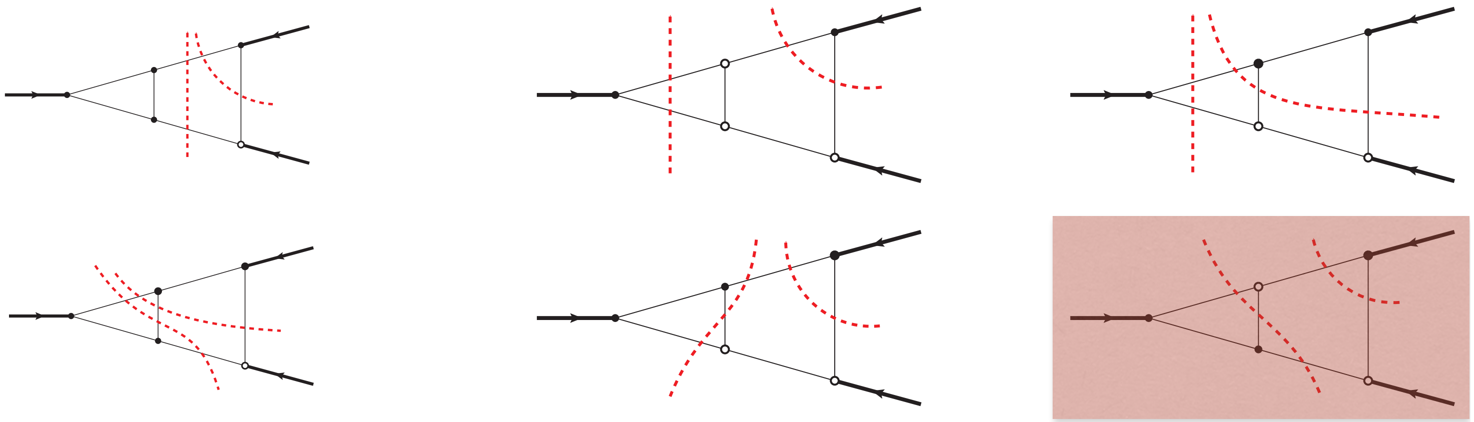
- $p^2{}_2$-channel cuts

$$\mathrm{Cut}_{p_2^2} T_L(p_1^2, p_2^2, p_3^2) = -\frac{2\pi (p_1^2)^{-2}}{(1-z)(1-\bar{z})(z-\bar{z})} \Big\{ 3\big[\mathrm{Li}_3(\bar{z}) - \mathrm{Li}_3(z)\big] + \big(\ln(z\bar{z}) - i\pi\big)\big[\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})\big] \Big\}$$

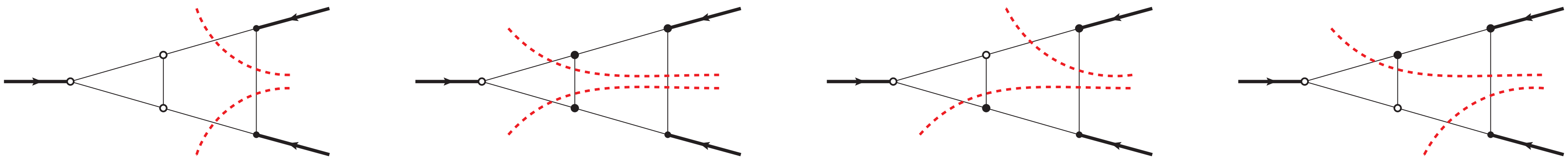
$$\boxed{\begin{aligned} \mathrm{Cut}_{p_2^2} T_L(p_1^2, p_2^2, p_3^2) &= -\mathrm{Disc}_{p_2^2} T_L(p_1^2, p_2^2, p_3^2) \\ &\cong -2\pi \left(p_1^2\right)^{-2} \frac{1}{(1-z)(1-\bar{z})(z-\bar{z})} \delta_{p_2^2} F(z, \bar{z}) \end{aligned}}$$

Example 2: three-mass three-point ladder

- $(p^2{}_3, p^2{}_1)$-channel cuts: $\delta_{p_3^2,\bar{z}} + \delta_{p_3^2,1-\bar{z}}$



- $(p^2{}_2, p^2{}_1)$-channel cuts: $\delta_{p_2^2,z} + \delta_{p_2^2,1-z}$



Example 2: three-mass three-point ladder

- $(p^2{}_3, p^2{}_1)$-channel cuts:

$$\boxed{\begin{aligned}\mathrm{Cut}_{p_3^2,p_1^2,R_{1,3}} T_L(p_1^2,p_2^2,p_3^2) =& [\mathrm{Disc}_{p_3^2,\bar{z}} + \mathrm{Disc}_{p_3^2,1-\bar{z}}] T_L(p_1^2,p_2^2,p_3^2)\\ \cong& (2\pi i)^2\,\Theta\,[\delta_{p_3^2,\bar{z}} + \delta_{p_3^2,1-\bar{z}}] T_L(p_1^2,p_2^2,p_3^2)\end{aligned}}$$

- $(p^2{}_2, p^2{}_1)$-channel cuts:

$$\boxed{\begin{aligned}\mathrm{Cut}_{p_2^2,p_1^2,R_{1,2}} T_L(p_1^2,p_2^2,p_3^2) =& [\mathrm{Disc}_{p_2^2,z} + \mathrm{Disc}_{p_2^2,1-z}] T_L(p_1^2,p_2^2,p_3^2)\\ \cong& (2\pi i)^2\,\Theta\,[\delta_{p_2^2,z} + \delta_{p_2^2,1-z}] T_L(p_1^2,p_2^2,p_3^2)\end{aligned}}$$

- **Triple cut**: unlike what happens with the one-loop triangle, this is as far as we can go with our cut conventions: after that we loose the connection to kinematic invariants.

Conclusions

- We provide a diagrammatic interpretation in terms of cuts of specific terms of the coproduct.

- We define a generalised set of cutting rules for multiple cuts, and conjecture relations between cuts computed with those rules, discontinuities across physical branch points and coproduct entries.

- That the relations we get are precise, including control of all minus sign, is a non-trivial check of our generalised cutting rules.

- Other cuts (that do not follow our rules) can reproduce deeper entries of the coproduct. However, the control of the signs and knowledge of the precise kinematic region is lost.

    - ex: the leading singularity can always be computed by the maximal cut up to an unknown overall normalisation.

- From the components of the coproduct we reproduce through our cut rules, we can reconstruct the full symbol of the uncut diagram through a well defined algorithm (relying on the integrability and the first entry condition).

Outlook

- How useful can cuts be to identify the correct variables for a given multi-leg diagram?

- What are the implications of our very strict cutting rules (dispersive representation, reconstruction of the symbol, generalisation of the largest time equation…)?

- How do this rules hold when we increase the number of loops and/or the number of legs?

- The coproduct provides a way to organise the information obtained from cuts. Can it be used to better interpret more general cuts (with complex kinematics)?