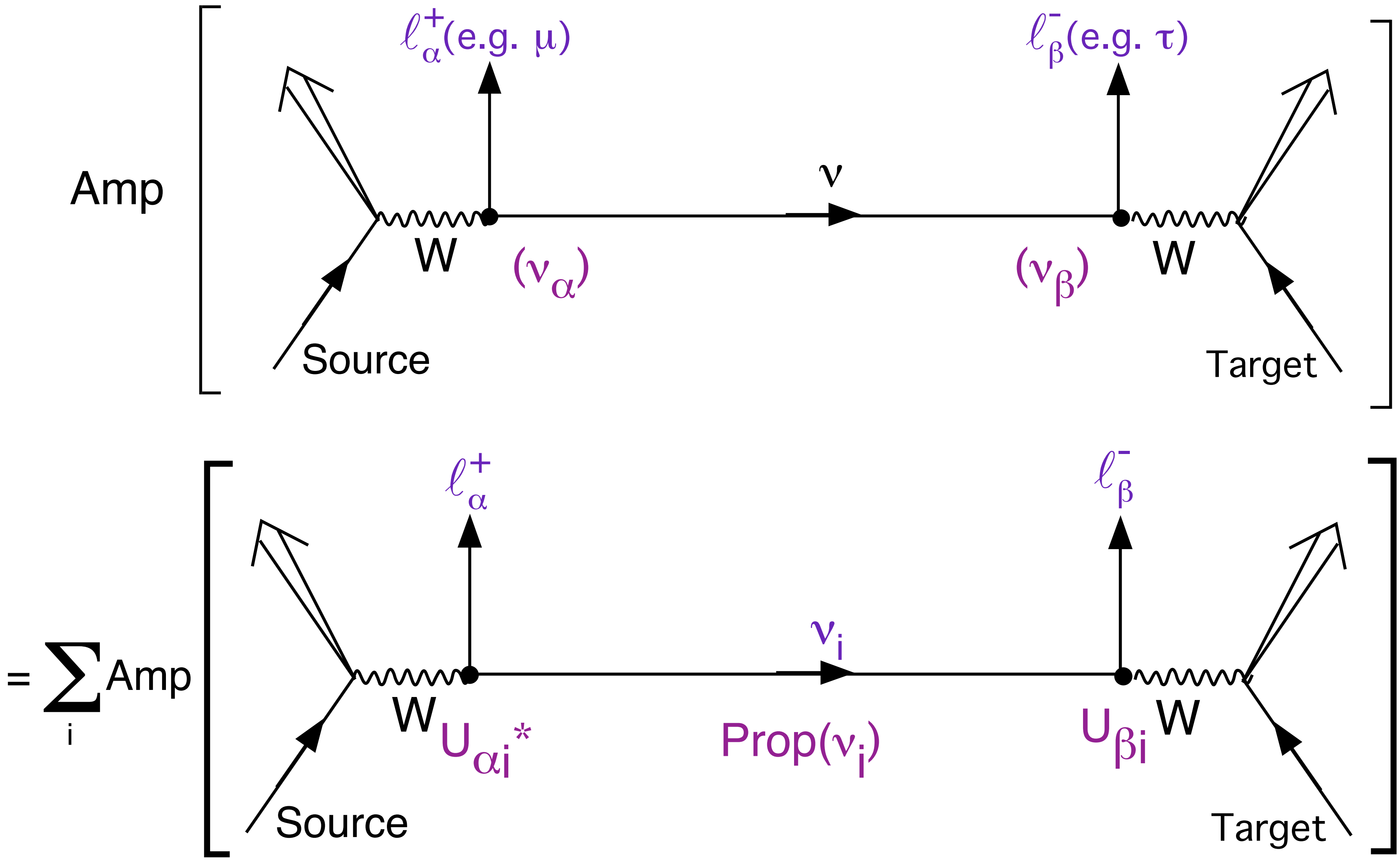
Neutrino Flavor Change (Oscillation) in Vacuum

(Approach of B.K. & Stodolsky)



$\text{Amp}\,[\nu_\alpha \rightarrow \nu_\beta] = \Sigma U_{\alpha i}^{*}\,\text{Prop}(\nu_i) U_{\beta i}$

What is Propagator (ν_i) $\equiv$ Prop(ν_i)?

In the ν_i rest frame, where the proper time is τ_i,

$$i\frac{\partial}{\partial\tau_i}|\nu_i(\tau_i)>= m_i|\nu_i(\tau_i)> \quad .$$

Thus,

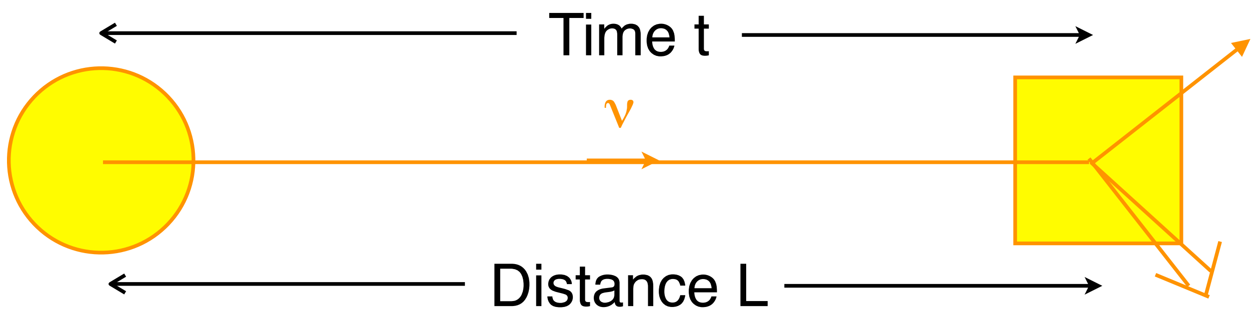
$$|\nu_i(\tau_i)>= e^{-im_i\tau_i}|\nu_i(0)> \quad .$$

Then, the amplitude for propagation for time τ_i

is —

$$\text{Prop}(\nu_i) \equiv < \nu_i(0)|\nu_i(\tau_i)>= e^{-im_i\tau_i} \quad .$$

In the laboratory frame —



The experimenter chooses L and t.

They are common to all components of the beam.

For each ν_i, by Lorentz invariance,

$$m_i \tau_i = E_i t - p_i L \,.$$

Neutrino sources are ~ constant in time.

Averaged over time, the

$e^{-iE_1t} - e^{-iE_2t}$ interference

is —

$$< e^{-i(E_1-E_2)t} >_t \; = \; 0$$

unless $E_2 = E_1$.

Only neutrino mass eigenstates with a common energy E are coherent.

(Stodolsky)

For each mass eigenstate ,

$$p_i = \sqrt{E^2 - m_i^2} \cong E - \frac{m_i^2}{2E} \quad .$$

Then the phase in the ν_i propagator $exp[-im_i\tau_i]$ is —

$$m_i\tau_i = E_i t - p_i L \cong Et - (E - m_i^2/2E)L$$

$$= \underbrace{E(t - L)}_{\text{Irrelevant overall phase}} + m_i^2 L/2E \quad .$$

What if the neutrino source is ***not*** constant in time?

The relative phase between two mass eigenstates,

$$\delta\phi(21) \equiv (E_2 t - p_2 L) - (E_1 t - p_1 L) \quad ,$$

is unchanged.

(Lipkin)

An approximation to the average speed of the ν_1 and ν_2 waves is

$$\bar{v} \equiv \frac{p_1 + p_2}{E_1 + E_2} \quad .$$

Then the travel time $t \cong L/\bar{v}$.

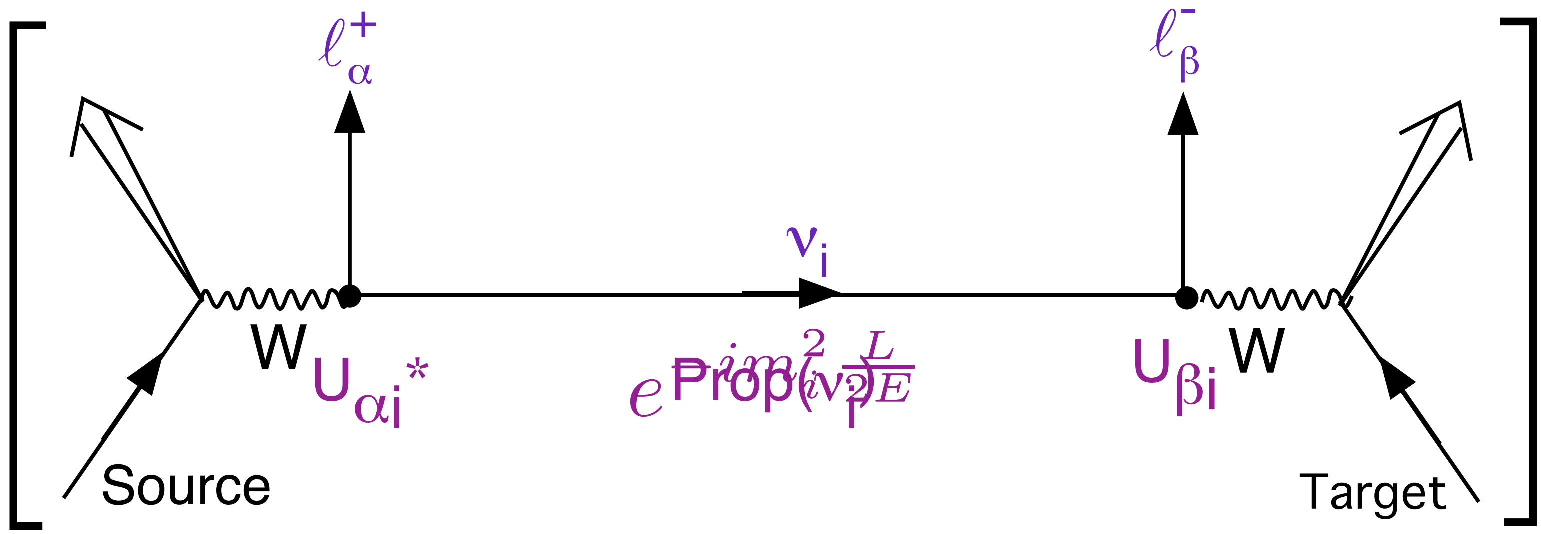
Thus,

$$\delta\phi(21) = (p_1 - p_2)L - (E_1 - E_2)t$$

$$\cong \frac{p_1^2 - p_2^2}{p_1 + p_2}L - \frac{E_1^2 - E_2^2}{p_1 + p_2}L \cong (m_2^2 - m_1^2)L/2E$$

Amp [$\nu_\alpha \rightarrow \nu_\beta$]

$= \sum_i \text{Amp}$



$$= \sum_i U^*_{\alpha i} e^{-i m_i^2 \frac{L}{2E}} U_{\beta i}$$

Probability for Neutrino Oscillation in Vacuum

$$P(\nu_\alpha \rightarrow \nu_\beta) = |\mathrm{Amp}(\nu_\alpha \rightarrow \nu_\beta)|^2 =$$

$$= \delta_{\alpha\beta} - 4 \sum_{i>j} \Re(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) \sin^2(\Delta m^2_{ij} \frac{L}{4E})$$

$$+2 \sum_{i>j} \Im(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) \sin(\Delta m^2_{ij} \frac{L}{2E})$$

$$\text{where } \Delta m^2_{ij} \equiv m^2_i - m^2_j$$

For Antineutrinos –

We assume the world is CPT invariant.

Our formalism assumes this.

$$P(\overline{\nu_\alpha} \to \overline{\nu_\beta}) \stackrel{CPT}{=} P(\nu_\beta \to \nu_\alpha) = P(\nu_\alpha \to \nu_\beta; U \to U^*)$$

Thus,

$$P(\overset{(-)}{\nu}_\alpha \to \overset{(-)}{\nu}_\beta) =$$

$$= \delta_{\alpha\beta} - 4 \sum_{i>j} \Re(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) \sin^2(\Delta m^2_{ij} \frac{L}{4E})$$

$$\underset{(-)}{+} 2 \sum_{i>j} \Im(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) \sin(\Delta m^2_{ij} \frac{L}{2E})$$

A complex U would lead to the CP violation
$P(\overline{\nu_\alpha} \to \overline{\nu_\beta}) \neq P(\nu_\alpha \to \nu_\beta)$.

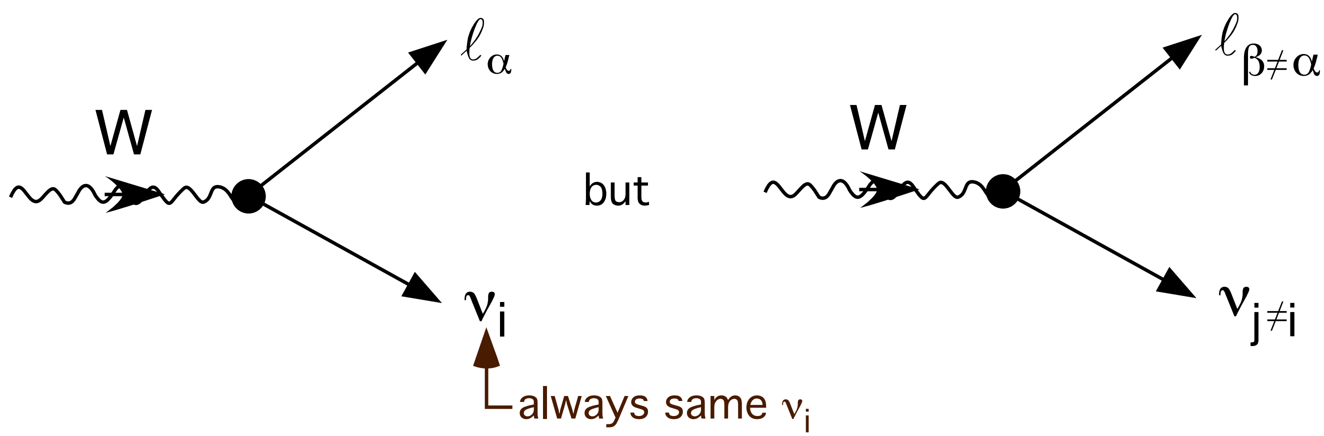
— Comments —

1. If all $m_i = 0$, so that all $\Delta m_{ij}^2 = 0$,

$$P(\overset{(-)}{\nu}_\alpha \rightarrow \overset{(-)}{\nu}_\beta) = \delta_{\alpha\beta}$$

Flavor *change* $\Rightarrow$ ν Mass

2. If there is no mixing,



$$\Rightarrow U_{\alpha i}U_{\beta\neq\alpha,i} = 0\text{, so that } P(\overset{(-)}{\nu}_\alpha \rightarrow \overset{(-)}{\nu}_\beta) = \delta_{\alpha\beta}.$$

Flavor *change* $\Rightarrow$ Mixing

3. One can detect ($\nu_\alpha \rightarrow \nu_\beta$) in two ways:

See $\nu_{\beta\neq\alpha}$ in a ν_α beam (Appearance)

See some of known ν_α flux disappear (Disappearance)

4. Including $\hbar$ and c

$$\Delta m^2 \frac{L}{4E} = 1.27 \Delta m^2(\mathrm{eV}^2) \frac{L(\mathrm{km})}{E(\mathrm{GeV})}$$

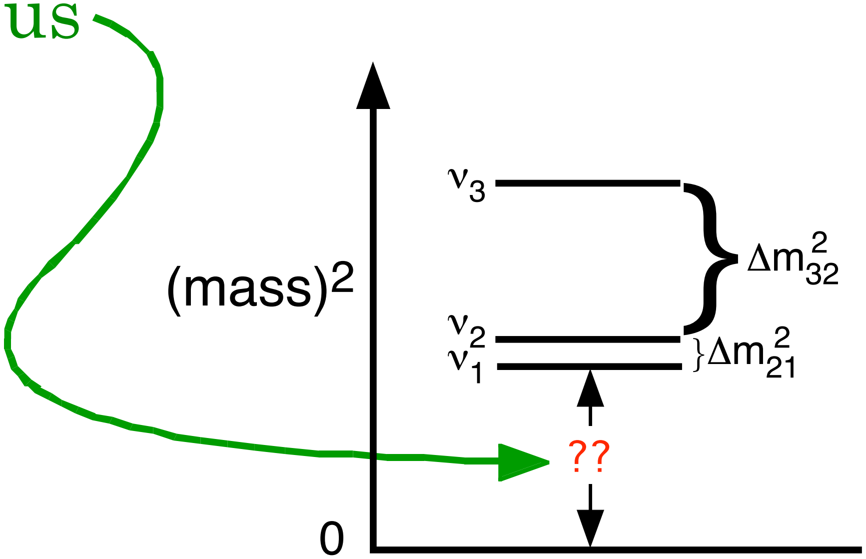
$\sin^2[1.27 \Delta m^2 (\mathrm{eV})^2 \frac{L(\mathrm{km})}{E(\mathrm{GeV})}]$ becomes appreciable when its argument reaches $\mathcal{O}(1)$.

An experiment with given L/E is sensitive to

$$\Delta m^2(\mathrm{eV}^2) \gtrsim \frac{E(\mathrm{GeV})}{L(\mathrm{km})} \quad .$$

5. Flavor change in vacuum oscillates with L/E. Hence the name "neutrino oscillation". {The L/E is from the proper time τ.}

6. P $(\overset{(-)}{\nu}_\alpha \rightarrow \overset{(-)}{\nu}_\beta)$ depends only on squared-mass splittings. Oscillation experiments cannot tell us



7. Neutrino flavor change does not change the total flux in a beam.

It just redistributes it among the flavors.

$$\sum_{\text{All } \beta} P(\overset{(-)}{\nu}_\alpha \rightarrow \overset{(-)}{\nu}_\beta) = 1$$

But some of the flavors β≠α could be sterile.

Then some of the *active* flux disappears:

$$\phi_{\nu_e} + \phi_{\nu_\mu} + \phi_{\nu_\tau} < \phi_{\text{Original}}$$

8. Assuming all coherent ν_i in a beam have a common momentum p, rather than a common energy E, is a harmless error.

This assumption leads to the same $P(\overset{(-)}{\nu}_\alpha \rightarrow \overset{(-)}{\nu}_\beta)$.

Important Special Cases

Three Flavors

For $\beta \neq \alpha$,

$$e^{-im_1^2 \frac{L}{2E}} \mathrm{Amp}^*(\nu_\alpha \to \nu_\beta) = \sum_i U_{\alpha i} U^*_{\beta i} e^{im_i^2 \frac{L}{2E}} e^{-im_1^2 \frac{L}{2E}}$$

$$= U_{\alpha 3} U^*_{\beta 3} e^{2i\Delta_{31}} + U_{\alpha 2} U^*_{\beta 2} e^{2i\Delta_{21}} \underbrace{- (U_{\alpha 3} U^*_{\beta 3} + U_{\alpha 2} U^*_{\beta 2})}_{\mathrm{Unitarity}}$$

$$= 2i[U_{\alpha 3} U^*_{\beta 3} e^{i\Delta_{31}} \sin \Delta_{31} + U_{\alpha 2} U^*_{\beta 2} e^{i\Delta_{21}} \sin \Delta_{21}]$$

$$\text{where } \Delta_{ij} \equiv \Delta m^2_{ij} \frac{L}{4E} \equiv (m_i^2 - m_j^2) \frac{L}{4E} \quad .$$

$$P(\overset{(-)}{\nu}_\alpha \rightarrow \overset{(-)}{\nu}_\beta) = \left| e^{-im_1^2 \frac{L}{2E}} \mathrm{Amp}^* (\overset{(-)}{\nu}_\alpha \rightarrow \overset{(-)}{\nu}_\beta) \right|^2$$

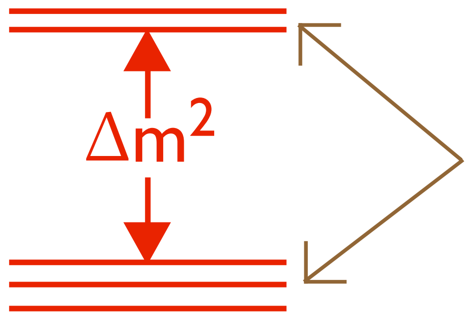
$$= 4[|U_{\alpha 3} U_{\beta 3}|^2 \sin^2 \Delta_{31} + |U_{\alpha 2} U_{\beta 2}|^2 \sin^2 \Delta_{21}$$

$$+2|U_{\alpha 3} U_{\beta 3} U_{\alpha 2} U_{\beta 2}| \sin \Delta_{31} \sin \Delta_{21} \cos(\Delta_{32} \underset{(-)}{+} \delta_{32})] \quad .$$

$$\text{Here } \delta_{32} \equiv \arg(U_{\alpha 3} U^*_{\beta 3} U^*_{\alpha 2} U_{\beta 2})\,, \text{ a CP} - \text{violating phase.}$$

Two waves of different frequencies,
and their $\cancel{\text{CP}}$ interference.

When One Big Δm^2 Dominates



These splittings are invisible if $\Delta m^2 \dfrac{L}{E} = \mathcal{O}(1)$.

For $\beta \neq \alpha$,

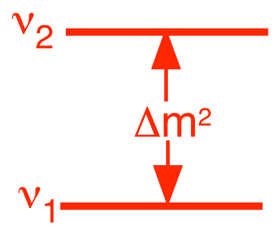
$$P(\overset{(-)}{\nu}_\alpha \rightarrow \overset{(-)}{\nu}_\beta) \cong S_{\alpha\beta} \sin^2(\Delta m^2 \frac{L}{4E}) \; ; \; S_{\alpha\beta} \equiv 4 \left| \sum_{i \text{ Clump}} U^*_{\alpha i} U_{\beta i} \right|^2 .$$

For no flavor change,

$$P(\overset{(-)}{\nu}_\alpha \rightarrow \overset{(-)}{\nu}_\alpha) \cong 1 - 4T_\alpha(1 - T_\alpha) \sin^2(\Delta m^2 \frac{L}{4E}) \; ; \; T_\alpha \equiv \sum_{i \text{ Clump}} |U^*_{\alpha i}|^2 .$$

"i Clump" is a sum over only the mass eigenstates on one end of the big gap Δm^2.

When There are Only Two Flavors and Two Mass Eigenstates



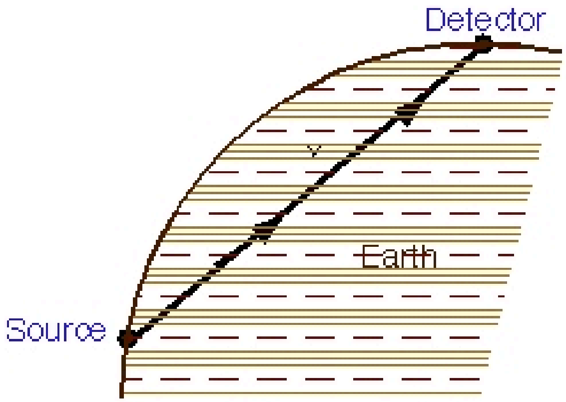
$$U = \begin{array}{cc} & \begin{array}{cc} \nu_1 & \nu_2 \end{array} \\ \begin{array}{c} \nu_\alpha \\ \nu_\beta \end{array} & \left[\begin{array}{cc} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{array} \right] \end{array} \quad ; \quad S_{\alpha\beta} = 4T_\alpha(1 - T_\alpha) = \sin^2 2\theta$$

Mixing angle

For $\beta \neq \alpha$, $\quad P(\overset{(-)}{\nu}_\alpha \leftrightarrow \overset{(-)}{\nu}_\beta) = \sin^2 2\theta \sin^2(\Delta m^2 \frac{L}{4E}) \quad .$

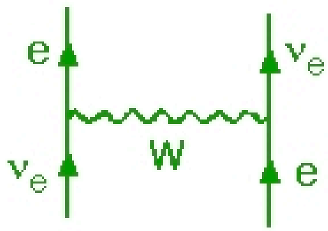
For no flavor change, $P(\overset{(-)}{\nu}_\alpha \rightarrow \overset{(-)}{\nu}_\alpha) = 1 - \sin^2 2\theta \sin^2(\Delta m^2 \frac{L}{4E})$.

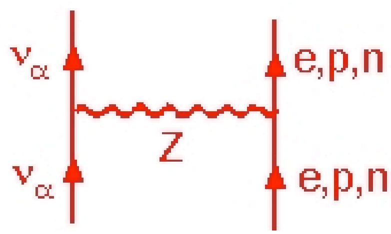
Neutrino Flavor Change in Matter



Coherent forward scattering from ambient matter can have a big effect.

Interaction





Interaction Potential Energy

$$V_W = +\sqrt{2}G_F N_e \qquad (-\text{ for } \overline{\nu_e})$$

#e/vol

$$V_Z = -\frac{\sqrt{2}}{2}G_F N_n \qquad (+\text{ for } \overline{\nu_\alpha})$$

#n/vol

Neutrino propagation in matter is conveniently treated via a Schrödinger Equation:

Matrix in flavor space

$$i\frac{\partial}{\partial t}\nu(t) = H\,\nu(t)$$

Multi-component in flavor space

To illustrate, we describe the case —

When Only Two Neutrinos Count

Amp. to be a ν_e

$$\nu(t) = \begin{bmatrix} f_e(t) \\ f_\mu(t) \end{bmatrix} \quad ; \quad H = \begin{array}{c c} & \begin{array}{cc} \nu_e & \nu_\mu \end{array} \\ \begin{array}{c} \nu_e \\ \nu_\mu \end{array} & \begin{bmatrix} & \\ & \end{bmatrix} \end{array}$$

Amp. to be a ν_μ

A 2x2 matrix in ν_e-ν_μ space

$$i\frac{\partial}{\partial t}\left[\begin{array}{c} f_e(t) \\ f_\mu(t) \end{array}\right] = \left[\begin{array}{ccc} & & \\ & H & \\ & & \end{array}\right]\left[\begin{array}{c} f_e(t) \\ f_\mu(t) \end{array}\right]$$

In Vacuum:

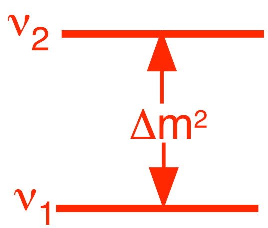
$$< \nu_\alpha | H | \nu_\beta > = < \sum_i U^*_{\alpha i} \nu_i | H | \sum_j U^*_{\beta j} \nu_j > = \sum_j U_{\alpha j} U^*_{\beta j} \sqrt{p^2 + m_j^2}$$

Momentum of the beam

In flavor change, only relative phases, hence relative energies, matter.

$\therefore$ In H, any multiple of the Identity Matrix I may be omitted.

In Vacuum



$$U = \begin{array}{c c} & \begin{array}{cc} \nu_1 & \nu_2 \end{array} \\ \begin{array}{c} \nu_e \\ \nu_\mu \end{array} & \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \end{array} \;\; ; \qquad \begin{array}{l} \nu_e = \nu_1 \cos\theta + \nu_2 \sin\theta \\ \nu_\mu = \nu_1(-\sin\theta) + \nu_2 \cos\theta \end{array}$$

It follows that, omitting a piece $\propto$ I,

$$H_{\text{Vac}} = \frac{\Delta m^2}{4E} \begin{bmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} \quad .$$

With Schrödinger's Equation, this gives the usual $P(\nu_e \to \nu_\mu)$.

The eigenvalues of H_{Vac} are —

$$\pm\frac{\Delta m^2}{4E} \equiv \pm\lambda \quad .$$

With $c \equiv \cos\theta$, $s \equiv \sin\theta$,

$$\nu_e = \nu_1 c + \nu_2 s \xrightarrow{t} \nu(t) = \nu_1 c e^{i\lambda t} + \nu_2 s e^{-i\lambda t}$$

$$P(\nu_e \to \nu_\mu) = \ |<\nu_\mu|\nu(t)>|^2 = |sc(-e^{i\lambda t} + e^{-i\lambda t})|^2$$

$$= \sin^2 2\theta \sin^2(\Delta m^2 \frac{L}{4E})$$

In Matter

$$H_M = H_{\mathrm{Vac}} + \quad V_W \begin{array}{cc} \begin{matrix}\nu_e & \nu_\mu\end{matrix} & \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{matrix}\nu_e \\ \nu_\mu\end{matrix} \end{array} + V_Z \underbrace{\begin{array}{cc} \begin{matrix}\nu_e & \nu_\mu\end{matrix} & \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{matrix}\nu_e \\ \nu_\mu\end{matrix} \end{array}}_{\propto \text{ I, so drop}}$$

$$H_M = H_{\mathrm{Vac}} + \frac{V_W}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{V_W}{2} \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$H_M = \frac{\Delta m^2}{4E} \begin{bmatrix} -(\cos 2\theta - x) & \sin 2\theta \\ \sin 2\theta & \cos 2\theta - x \end{bmatrix} ,$$

with $\quad x \equiv \dfrac{V_W/2}{\Delta m^2/4E} = \dfrac{2\sqrt{2} G_F N_e E}{\Delta m^2} \quad .$

The Effective Splitting and Mixing in Matter

If we define —

$$\Delta m_M^2 \equiv \Delta m^2 \sqrt{\sin^2 2\theta + (\cos 2\theta - x)^2}$$

and

$$\sin^2 2\theta_M \equiv \frac{\sin^2 2\theta}{\sin^2 2\theta + (\cos 2\theta - x)^2} \quad ,$$

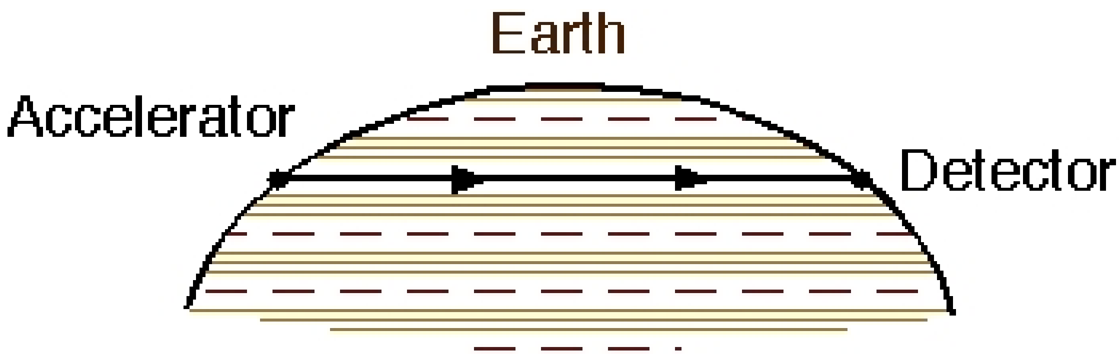
then

$$H_M = \frac{\Delta m_M^2}{4E} \begin{bmatrix} -\cos 2\theta_M & \sin 2\theta_M \\ \sin 2\theta_M & \cos 2\theta_M \end{bmatrix} \quad .$$

This is H_{Vac} with $(\Delta m^2, \theta) \rightarrow (\Delta m_M^2, \theta_M)$.

Thus, Δm_M^2 and θ_M are the effective splitting and mixing angle in matter.

Travel Through the Earth



The matter density encountered en route is ~ constant.

Thus, H_M is position-independent, just like H_{Vac}.

Therefore, in the earth (but not too deep),

$$P_M(\nu_e \rightarrow \nu_\mu) = \sin^2 2\theta_M \sin^2(\Delta m_M^2 \frac{L}{4E})$$

In matter

The Size and Consequence of the Matter Effect

The matter effect depends on —

$$x = \frac{2\sqrt{2}G_F N_e E}{\Delta m^2} \propto E \quad .$$

The denominator contains a Sign

In the earth's mantle, for $|\Delta m^2| = |\Delta m^2(\text{atmospheric})|$ $\cong 2.7 \times 10^{-3}$ eV^2,

$$|x| \simeq \frac{E}{12\text{GeV}} \quad .$$

Since $V_W(\bar{\nu}) = -V_W(\nu)$, $x(\bar{\nu}) = -x(\nu)$.

Thus $\overline{\Delta m_M}^2 \neq \Delta m_M^{\,2}$ and $\sin^2 2\bar{\theta}_M \neq \sin^2 2\theta_M$.

The matter effect causes an asymmetry between $\bar{\nu}$ and ν oscillation. This must be separated from the genuine $\cancel{CP}$ asymmetry.

The MSW Effect

Since —

$$\sin^2 2\theta_M = \frac{\sin^2 2\theta}{\sin^2 2\theta + (\cos 2\theta - x)^2} \quad ,$$

even a tiny vacuum mixing $\sin^2 2\theta$ can be amplified into a near-maximal in-matter mixing $\sin^2 2\theta_M$ if

$$x \cong \cos 2\theta \quad .$$

This is the "resonant" version of the —

Mikheyev Smirnov Wolfenstein Effect.

This is *NOT* what happens in the sun!