Captain Joseph Kittinger free falls from 31km above the Earth

Chapter 16

The Equivalence Principle

It seems up to this point that inertial frames play a preferred role in the formulation of physical theories and you may feel uneasy with this state of affairs. If so, you would be in good company - Einstein, motivated by the observation that an accelerating frame appears to reproduce the sensation of a gravitational field (§5), proposed an extension of the principle of relativity which we will soon discuss. Pursuit of this train of thought ultimately lead Einstein to his theory of "General Relativity", (GR), and that is where we too are headed. This theory is viewed by many as a pinnacle of mankind's achievements - the scientific equivalent of Beethoven's 9th Symphony. It is unfortunate that a high level of mathematical machinery is required to do justice to the subject. We will largely avoid that machinery, but nevertheless the remainder of this course we will try to explore the main ideas of GR, starting here with the "Principle of Equivalence".

16.1 Inertial Mass and Gravitational Mass

Let us begin with a closer look at the concept of mass. Two distinct types of mass enter into Newtonian mechanics. There is the mass that appears in the Second Law:

$$\underline{f} = m_I \underline{a} \tag{16.1}$$

usually referred to as *inertial mass*. Irrespective of the nature of the particular force acting, this describes the inertia of a body (i.e. its resistance to changing its state of motion).

In Newton's Law of Gravity, the masses which appear are *gravitational masses*:

$$\underline{f} = -G\frac{M_G m_G}{r^2}\hat{\underline{r}}. \tag{16.2}$$

Gravitational mass is analogous to the electric charge of a particle in electrostatics and there is no *a priori* reason why either quantity should have any connection with m_I. Yet gravitational mass, responsible for the particular force we call gravitation, is observed to be proportional to the inertial mass entering the general force law. (In suitable units, they are equal, $m_G = m_I = m$.) Because we are familiar with Newton's laws we tend not to think about this but it really is remarkable. The equality of inertial and gravitational mass, sometimes referred to as *the weak equivalence principle*, has been verified experimentally with increasing precision since the first experiment by Eötvös in 1889, and is now known to hold to better than 1 part in 10^{12}. It is an experimental observation which must be regarded as an accident or coincidence since neither Newton's second or his gravitational law gives any hint of the deep connection with the other.

16.2 Galileo's Principle and Einstein's Thought Experiments

Combining Eq. 16.1-16.2 we get $|\underline{a}| = GM/r^2$ and so weak equivalence implies "Galileo's Principle":

All *objects experience the same acceleration in a given gravitational field.*

To investigate the consequences of Galileo's principle, Einstein proposed a series of thought experiments involving an observer confined within a box. The box serves to deny the observer any information from outside. He has no windows to look out of or wi-fi hotspots but can relieve the boredom of his captivity by performing rudimentary experiments, and only by this means can he study his state of motion.

The first such experiment is rather dull. The observer takes keys from his pocket and releases them from rest, seeing them fall to the floor as expected. The next day the observer feels strange, as if floating. The experiment is repeated and the keys are found to hang motionless in mid air. The observer suspects his box is now floating in space, far from any detectable gravitational field. In this scenario the rest frame of box and observer is inertial and N1 should hold, and indeed it does for the keys. Then the observer's thoughts turn a little darker. Mindful of Galileo's principle, he realises that if his box were in free-fall towards the surface of the Earth (probably impling imminent death) the keys would fall with the same acceleration as he and so would appear to hang in mid air, just as he observed. The observer can think of no way that he could distinguish between free-fall under gravity and floating in space with no gravitational fields at all!

Now let's place Einstein's box in deep space and cause it to accelerate at 9.8 $\mathrm{m\,s^{-2}}$ by attaching a rocket. The force from the rocket is transmitted to the observer by the floor of the box. Einstein pointed out that the observer would feel very much at home in this exotic environment. In particular his keys would once more fall to the floor with an acceleration of 9.8 $\mathrm{m\,s^{-2}}$, just like at the Earth's surface.

16.3 The Strong Equivalence Principle

Should Galileo's principle apply to the photon? Eq. 16.2 suggests the photon, having no mass, should not feel gravity at all. Einstein believed the apparent indistinguishability of floating in deep space and freely falling under gravity is the fundamental issue, and he re-stated the principle of equivalence thus:

All local, freely falling, non-rotating laboratories are fully equivalent for the performance of all physical experiments.

By 'freely falling laboratory' is meant one which is accelerated by the local gravitational field without any restraining forces. Such a freely falling laboratory is said to constitute a *local inertial frame* or LIF. It is as if the acceleration can cancel out the gravitational effects. Similarly, acceleration can generate the sensation of gravity where there is no gravitating mass.[1]

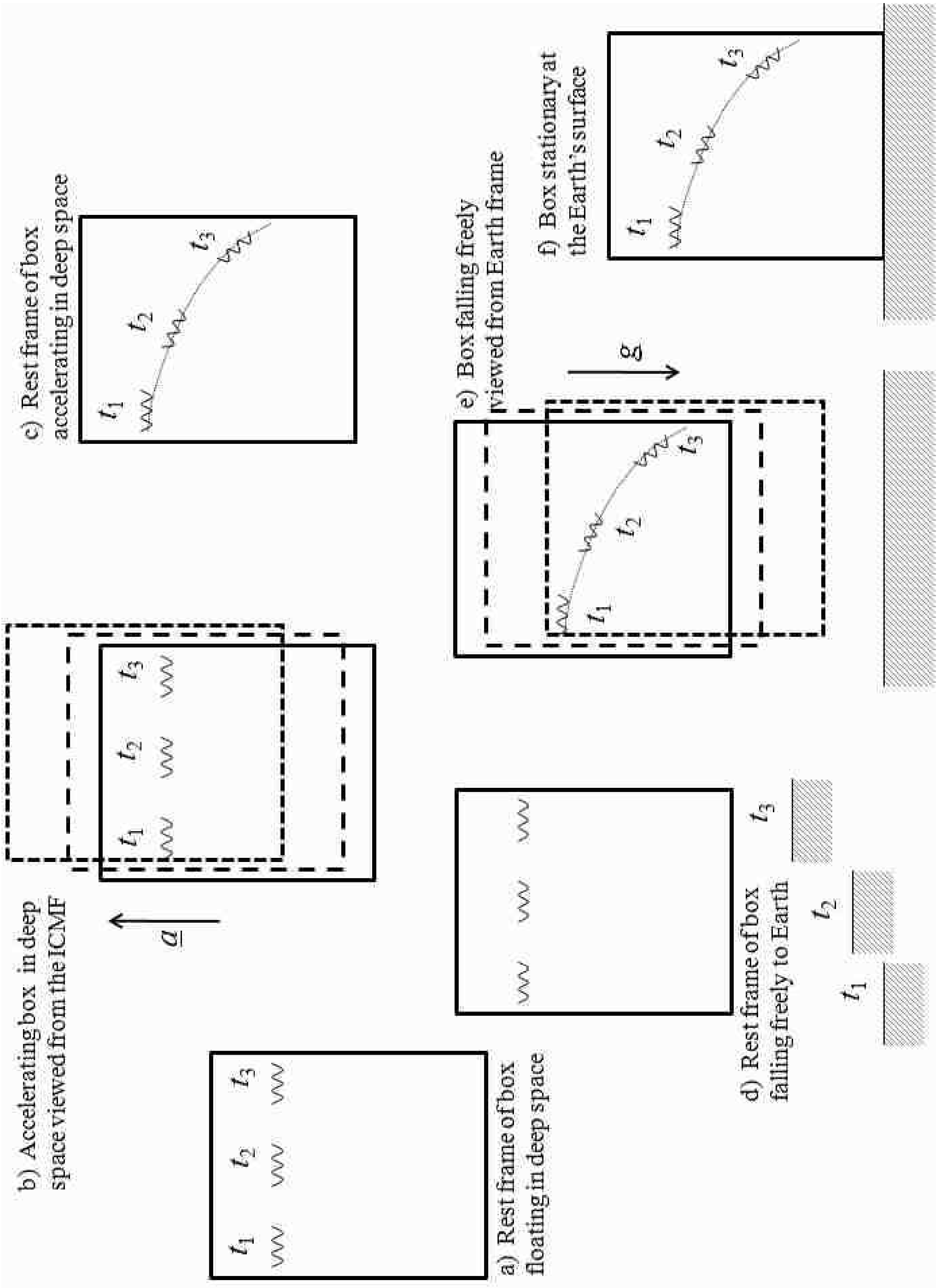
We refer to the boxed text above as Einstein's "Strong" Equivalence Principle (SEP) but it is not yet clear that this statement implies anything beyond Galileo's Principle. There is a clue in the phrase "all physical experiments". The SEP encompasses the behaviour of light in a gravitational field.

16.4 Predictions of the SEP

Suppose Einstein's box is floating in deep space and a light signal is emitted horizontally across the box. The observer will find the light travels horizontally of course, as shown in part a) of the figure below. Now we arrange for a second photon to be emitted in the same manner at the precise moment the box begins to accelerate at rate $\underline{a} = -\underline{g}$, where $\underline{g}$ is the gravitational acceleration at the Earth's surface. Using the ICMF (instantaneous co-moving inertial frame) at the moment the photon sets off, the path of the photon is unchanged (see part b) of the figure). Although the box is initially at rest in this frame, it moves upwards during the transit of the photon (by an amount $\frac{1}{2}at^2$), and so in the rest frame of the box (see part c)) the path of the light appears to be curved.[2] So far all this is in no way controversial. However, according to Einstein's SEP *gravity should also curve the path of the light beam* when the box is not accelerating, as in part f). This is a new idea.

[1] A cautionary remark: Two particles initially on opposite sides of a box in free fall will move towards each other very slightly since each is heading towards the centre of the Earth. This does not occur for two particles in an inertial box in deep space. Thus the equivalence of gravity and acceleration strictly only applies *locally*. The freely falling box has to be small enough: we require different "Local Inertial Frames" (LIFs) at different points in space. (The goal of General Relativity is to find a way to patch together the differing LIFs in a way that takes account of the distribution of gravitating matter in the universe.)

[2] The accelerating observer might interpret the curved trajectory as the effect of a "fictitious acceleration", after all he would be able to feel the acceleration of his box.

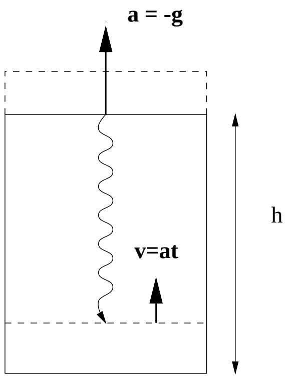

a) Rest frame of box floating in deep space
b) Accelerating box in deep space viewed from the ICMF
c) Rest frame of box accelerating in deep space
d) Rest frame of box falling freely to Earth
e) Box falling freely viewed from Earth frame
f) Box stationary at the Earth's surface
a
g
t1
t2
t3

Consider also the case shown in part d). Here the box is allowed to begin falling freely in the Earth's gravitational field at the moment that a photon sets off across the box. We might expect the massless photon to ignore the gravitational field while the box starts to drop, implying that the photon would appear to the observer in the box to curve upwards. However the SEP requires that the observer sees the photon travelling straight across the box, just as in part a). The photon is predicted to fall with the observer!

Had the photon been fired from the ceiling of our accelerating box, as in the diagram below, a different phenomenon would have emerged. To complete its journey to the floor the light takes time t_H, given by $ct_H + \frac{1}{2}gt_H^2 = H$, where H is the height of the box. In time t_H the box acquires a speed

$$v_H = gt_H = c\left[\left(1 + 2gH/c^2\right)^{1/2} - 1\right] \approx \frac{gH}{c} \tag{16.3}$$

vertically upwards, as judged by an ICMF observer. An observer lying on the floor of the box is moving in the rest frame of the emitter, and so observes a Doppler shift.



When source and observer move apart at speed v the observed frequency, $\nu_{\rm obs}$, is related to the frequency of the source in its rest frame, $\nu_{\rm source}$, by

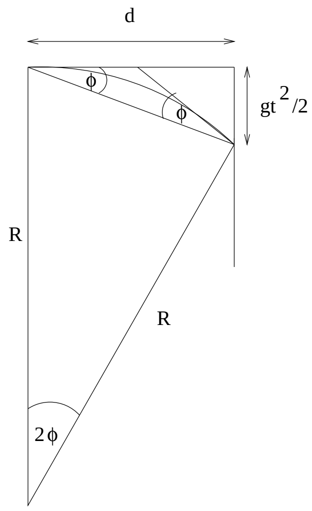
$$\nu_{\rm obs} = \nu_{source}\sqrt{\frac{c-v}{c+v}} \approx \nu_{source}(1 - v/c). \tag{16.4}$$

In this case we have a blue shift, and the fractional change in frequency is

$$\frac{\Delta\nu}{\nu} \equiv \frac{\nu_{\rm obs} - \nu_{\rm source}}{\nu_{\rm source}} = \frac{v_H}{c} = \frac{gH}{c^2}. \tag{16.5}$$

By the SEP, gravity should also give rise to a Doppler effect *without the need for relative motion between source and observer.*

16.5 Testing Strong Equivalence



A photon crossing a box of width d and acceleration g drops an amount $gd^2/2c^2$, and we can evaluate the corresponding (approximate) radius of curvature, R, of the light's path (which we assume is an arc of a circle of radius R). For small deflections, $2R\phi \simeq d$. The angle ϕ is the angle between the tangent to the path and the chord, as indicated in the figure. Thus

$$\tan\phi = \frac{gt^2}{2d} = \frac{gd}{2c^2} \simeq \phi \simeq \frac{d}{2R} \quad \Rightarrow \quad R = \frac{c^2}{g}. \tag{16.6}$$

At the surface of the Earth the deviation of light by the Earth's gravitational field is too small to detect ($R \approx 10^{16}$ m). However, during a solar eclipse in 1919 Eddington observed $\phi \sim 0.001°$ for light rays deviated by the sun (which provides us with a spectacularly greater gravitational field), in excellent agreement with predictions based on the above arguments.

The first experimental measurement of the gravitational Doppler redshift was made in 1960 by Pound & Rebka, and improved by Pound & Snider in 1965. The experiment used the Mössbauer effect, the recoil-free emission of a gamma ray from a nucleus in a crystal,to give radiation of very precisely-defined frequency. They measured the frequency difference produced when a photon 'climbs' a distance $H = 22.5$ m, and found a redshift of magnitude $(2.57 \pm 0.26) \times 10^{-15}$, to be compared with the theoretical prediction of 2.46×10^{-15}. Einstein's SEP appears to be correct.

16.6 Curvature

The discussions above show that the trajectory of light in the presence of gravitation corresponds not to a straight line, but to a curve. General Relativity attributes this phenomenon to the curvature of space itself.

The movement of light is comparable to that of a train compelled to travel straight ahead on its tracks, but with the track itself being curved.

Since this is such a hard idea let's build our thoughts slowly and by analogy. Although the mathematics of tensors is required if this subject is to be explored in any detail, here we will try to get to the main ideas with a minimum of mathematics by appealing instead to physical arguments.

16.6.1 Curved 2D Surfaces

Consider primitive insects that have a two dimensional (2D) existence. We assume that our insects cannot see, have limited imagination but they can move around. They have no way of leaving their 2D worlds, or of even knowing that anything more exists. We can imagine a number of different 2D worlds, such as a flat plane, a long cylinder or a spherical ball. A less obviouis 2D world is known as the "hotplate". From our 3D point of view, "hotplate world" is a 2D plane in which the temperature is a minimum at one particular point and increases with distance from that point. (Our insects have no sense of temperature.)

Now suppose that our insects learn how to draw lines and measure lengths. They soon become curious about geometry. Perhaps the most fundamental idea in geometry is that of the straight line: the shortest distance between two points. Form our point of view, the insect on the plane draws "correct" straight lines, but the insects on the sphere and cylinder are unaware that their lines appear curved to us. The insect on the hotplate is also unaware of a problem with his lines which arises on account of the thermal expansion. The problem is that when the insect places his ruler in a hot area, it expands. (We assume that the ruler and the hotplate (and the insect for that matter) have the same coefficient of thermal expansion.) It follows that the straight line which indicates the shortest distance between any two points on the hotplate tends to be curved towards hotter regions.

It is important to appreciate that each insect can draw lines which are straight to him without any knowledge that there are 3D creatures (e.g. us) who might take a different view. But is there any way these insects can determine whether their worlds are curved? It turns out that the answer is "yes". Let's suppose that an insect on each of our 2D worlds decides to think more seriously about geometry and discovers the results of Euclid. In 2D Euclidean (i.e. "familiar") geometry:

- If one marks out a triangle and then sums up the internal angles then the answer is always 180°.
- If one travels in straight line for a particular distance, let's say 1 metre, and then turns 90° to left, and then one repeats this procedure three times, then one should return precisely to the start point.
- If one draws a circle (the locus of points which are equidistant from the chosen origin) then the length around the circumference should be 6.28 times the length of the radius.

Each of our insects can compare these theoretical results with the reality of their own worlds. In each case, small scale experiments will conform to the Euclidean ideal. (Locally Euclidean surfaces are known as *Riemannian*.) However, the insects on the sphere and the hotplate will observe discrepancies as the scale of their measurements (e.g. the size of their test circle) increases. The sphere and the hotplate worlds are said to have *intrinsic curvature*. The plane and, perhaps surprisingly, the cylinder have ideal Euclidean geometry. In simplistic terms we could say that *intrinsic curvature* requires some kind of distortion of the 2D plane, beyond simply embedding it in a 3D environment. It is intrinsic curvature that is relevant to Einstein's theorey of gravitation. From now on, when we say "curvature" we will mean "intrinsic curvature".

16.6.2 Curvature in 3D

We saw in the previous section that it is possible to test the curvature of your space using only local measurements made within it. It is not necessary to be somehow magically elevated to higher dimensionality than your own world, nor is it necessary to explore the full extent of your world, e.g. by circumnavigating the spherical surface. While it is rather hard to imagine how 3D space can be curved, we can apply local curvature tests, as we discussed above. In fact the concept of curvature is somewhat more complicated than in 2D: in 3D space the curvature depends not only on position but also on orientation (of our test circles and test triangles etc.) at that point. To sidestep this complexity we will consider what is effectively an average curvature by comparing the measured surface area A and radius r of a test spehere. Euclidean geometry predicts the radius of a sphere to be $\sqrt{A/4\pi}$. It turns out that the Earth has a radius which is approximately 1 mm longer than it should be for its surface area, while the Sun has an excess radius of about 1000 metres or so. Einstein claimed that this curvature is due to gravitation, and he proposed a law for the excess radius:

$$\sqrt{\frac{A}{4\pi}} - r = \frac{GM}{3c^2} \tag{16.7}$$

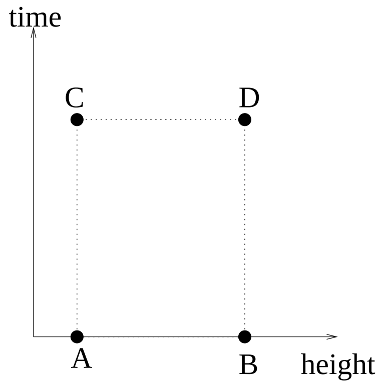
where G is Newton's gravitational constant, M is the mass (assumed uniformly spread) within a sphere of radius r. In everyday life we interpret the curvature as the gravitational force.

16.6.3 Curvature of Space-Time

At the end of §14 we saw a puzzle: it seemed that combining knowledge of gravitational energies, photon energies and Einstein's mass-energy idea leads to the breakdown of energy conservation. The law of energy conservation has proved so useful and robust that we will not abandon it now. Rather we should suspect that the missing energy sneaked past us somehow, but where?

The answer is now at hand. Eq. 16.5 indicates an increase in photon energy given by $h\nu \times gH/c^2$ when light drops through height H in a region where the gravitational acceleration is g. This is just the right amount of energy to balance the books in §14.3! GR offers an intriguing interpretation of gravitational Doppler effect: it arises because the clocks used to measure frequency run slower at low altitude where the gravity is stronger. It is not really the clocks that are having difficulty in the presence of gravity - it is time itself that appears to run at different rates at different heights. Heart rates etc. also slow down. The photon frequency measured by the gravitationally distorted time units comes out higher than the corresponding measurements at high altitude. In other words we have a *gravitational time dilation* factor of $1 - gz/c^2$.

After plotting the axes of a standard Minkowski diagram (space along one axis and time along the other) consider two particles at rest in some inertial frame. One is ten metres vertically above the other and there are no gravitational forces. We can label the space-time events which measure the positions of particle 1 and particle 2 as A and B, as shown in the Figure. Ten seconds later we locate the particles again and label these events C and D. The points ABDC form a perfectly good rectangle and we can say that Minkowski space-time is Euclidean (i.e. flat). However, when the two particles are at different heights in a gravitational field something goes wrong with the space-time geometry. The clocks attached to particle 1 and 2 no longer run at the same rate and so the lengths AC and BD are not equal. The gravitational field has rendered space-time curved.



What next?

In this chapter we have pursued a quasi-classical discussion, using Newtonian acceleration formulae etc. Yet we have seen that Special Relatvity causes us to review the Newtonian view of acceleration. And note that the quasi-classical discussion in this chapter cannot treat the "twin paradox", where one twin travels at very high speed, since ah/c^2 cannot then be a small parameter. Furthermore the other kind of time dilation, neglected here, comes into play. Similarly a clock flown around the globe in an aeroplane, or a muon travelling through the Earth's atmosphere, are each subjected to both SR and GR time dilation phenomena. The gravitational effect dominates for the plane which travels rather slowly compared to c, whereas the SR effect dominates for the fast-moving muon.

We need to incorporate SR into the discussion of this chapter. We did not yet speak much about the perspective of an accelerating *relativistic* observer. To learn more about gravity we will do so now.