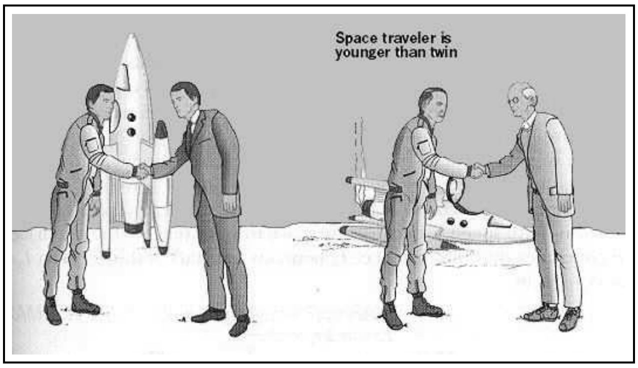

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Chapter 17

Accelerating Frames Revisted

17.1 Constant a_0 revisited

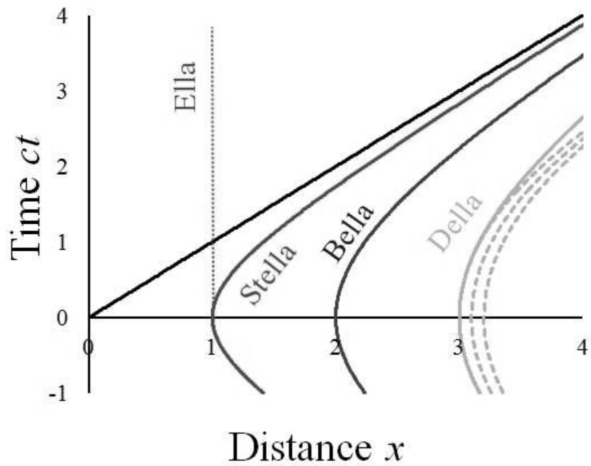
In §10.7 we figured out, using inertial co-ordinates, the speed and trajectory of an object undergoing constant proper acceleration a_0 in the x direction. Taking (purely for notational convenience) the position of the object to be c^2/a_0 at $t = 0$, we found

$$\frac{u}{c} = \frac{ct}{\sqrt{\rho^2 + c^2t^2}} \qquad x^2 - c^2t^2 = \rho^2 \tag{17.1}$$

where ρ is just short-hand for c^2/a_0. Thus the world-line of the object is a hperbola with the line $ct = x$ as an asymptote. Very much with the twin paradox in mind we compared the passing of inertial time with the proper time τ measured by a clock travelling with our accelerating object, finding

$$ct = \rho \sinh \frac{c\tau}{\rho} \qquad x = \rho \cosh \frac{c\tau}{\rho}. \tag{17.2}$$

In this way we were able to show how astronaut Stella, who accelerates away from and then back towards her twin Eartha, is able to keep track of her inertial co-ordinates and hence anticipate the age difference of the twins at their reunion. Stella's world line is illustrated here on the right (assuming $\rho = 1$).[1]



Though useful, the above does not mean we have considered an accelerating *frame of reference* within Special Relativity - that would imply the ability to specify space and time co-ordinates for *any* event, not just those on Stella's world-line. If we are to go beyond the quasi-Newtonian picture of Equivalence discussed in the previous chapter we must go further. We will see that even defining an appropriate space-time co-ordinate system is non-trivial, but we will be rewarded for our efforts. As well as obtaining more respectable derivations and accurate equations for the gravitational Doppler shift and spacetime curvature effects discussed in the previous chapter, we will see the emergence of a new and exotic phenomenon.

17.2 Stella's Rest Frame

Before proceeding, it is worth remembering how we constructed the gridlines of our Minkowsky diagram in Fig. 9.2. The vertical lines in our S frame grid can be thought of as the world-lines of an array of equally

[1]An acceleration of 9.5 $\mathrm{m\,s^{-2}}$ sets the unit of distance to 1 light-year.

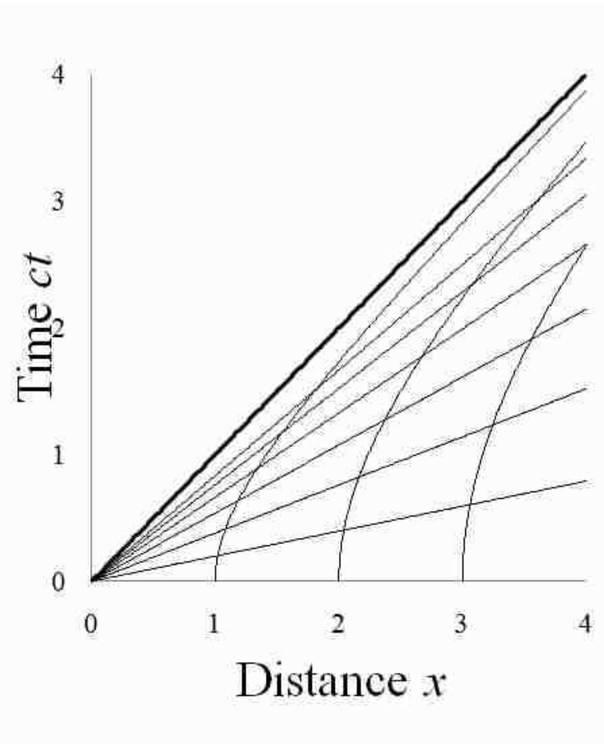
spaced objects. Meanwhile each horizontal line connects events that happen at the same time, i.e. they are lines of simultaneity. Using these two facts we were able to construct also a space-time grid for the (inertial) S' frame, also shown in Fig. 9.2. In the same spirit, Stella can map out distances in her accelerating frame using a succession of equally spaced observers. In the picture above we see that at $t = 0$ Stella is at rest at position $x = 1$ and we imagine that Bella is at $x = 2$ at this moment. At first glance it seems that, if Bella is to be stationary in Stella's rest frame, she must accelerate with the same proper acceleration as her friend so that their world lines would be parallel. Certainly two inertial observers with parallel world-lines would share a rest frame, but this is not the right approach here.

To see why, consider some particular choice of t, greater than 0, which we call t_1. We know Stella's speed and position at t_1 from Eq. 17.1 and we deduce $u_{Stella}(t_1)/c = ct_1/x_{Stella}(t_1)$. Now consider the inertial frame S', in standard configuration with S, whose x' axis passes through the point $(x_{Stella}(t_1), ct_1)$. We know (from §9.5) that the gradient of the x' axis is equal to v/c, where v is the speed of the S' frame. Thus $v/c = ct_1/x_{Stella}(t_1)$ and so the speed of S' matches Stella's instantaneous speed. S' is her ICMF at time t_1. If Bella and Stella have the same proper acceleration it is straightforward to show (using the LT) that the S' frame distance between the two increases with t_1, which is not suitable behaviour for two objects supposedly sharing a rest frame.

So how are we to construct Stella's rest frame? It is clear from Eq. 17.1b that *all* points on Stella's world-line are the same proper distance from the origin[2], and this would remain true whatever choice of ρ we made for Stella. We deduce that all observers (or particles, or clocks) satisfying Eq. 17.1b with *any* ρ (such as those shown in the illustration above) will be a fixed proper distance from Stella.[3] *They are stationary in Stella's rest frame.* But now for the surprise: a different choice of ρ means a different proper acceleration,[4] so:

All positions in Stella's rest frame are required to have different proper accelerations.

The closer their world-lines pass to the space-time origin, the smaller is their ρ parameter and the more severe is the acceleration needed to remain in Stella's rest frame.



We have seen that the spatial gridlines of Stella's rest frame are provided by the hyperbolic world-lines of accelerating observers sharing her asymptotes and centre. The temporal grid is provided by lines of simultaneity. Fortunately our discussion above already gives us all we need. In particular we found that the *inertial* frame whose x' axis intersects Stella's world-line at some particular time, is Stella's ICMF at that time. This x' axis is of course nothing other than the line of Simultaneity for $t' = 0$. Thus the time grid for Stella's rest frame is provided by the $ct' = 0$ lines of the succession of ICMFs Stella passes through. Using Eq. 17.2 the gradient of these lines, several of which are illustrated in the image to the left, can be expressed $\tanh c\tau/\rho$, where τ is Stella's proper time. In fact this expression should work for Bella and Della too, so we have another surprise: Along the line of simultaneity of the ICMF, different observers in Stella's rest frame measure a proper time proportional to their ρ parameter, hence:

Proper time runs at different rates in different positions in an acceleration field.

If the clocks of Stella, Bella and Della are synchronized at $t = 0$, they will drift out of synch, even though

[2] Remember that proper distance is a distance measured in a rest frame.
[3] Even though their separation changes with time according to an inertial observer.
[4] Just as points moving on circles of different radii have different accelerations.

they share a rest frame. The co-ordinate grid shown above is sometimes called “Rindler space”, and θ defined as τ/ρ the “Rindler time”. The requirement that $a_0 = c^2/x(0)$ is sometimes called “Born acceleration”.

One more observation for now: the proper acceleration becomes singular as $\rho \to 0$, but away from the singularity successive hyperbolae become almost parallel and almost parabolic, as shown by the dashed lines in the image on the previous page, provided we don't venture close to the light cone emanating from the space-time origin (where observer velocities are approaching c).

17.3 Free Particles in Rindler Space

Imagine an astronaut, Ella, falls out of Stella's rocket ship. In inertial frame S, Ella will continue at constant velocity with a straight world-line tangential to Stella's at the event of their separation. For simplicity, let's assume Ella is released at $ct = 0$, as illustrated above. In S we then have $x = 1$ for all subsequent times, with Ella crossing the event horizon (see §10.7) at $ct = 1$. Ella feels no particular physical sensation in so doing. She continues to receive light from Stella but receives no replies to messages she sends after $ct = 1$ simply because Stella will never receive them.

Ella sees Stella sail into the distance, but how does Ella's trajectory appear to Stella? In §5 we saw that free particles do not appear to move with uniform motion to an accelerating observer. We have seen that Stella's rest frame can be charted using the Rindler co-ordinates θ and ρ which are readily obtained from the inertial S frame co-ordinates, (x, ct) by inverting Eq. 17.2:

$$\rho = \sqrt{x^2 - c^2t^2} \qquad c\theta = \tanh^{-1}(ct/x). \quad (17.3)$$

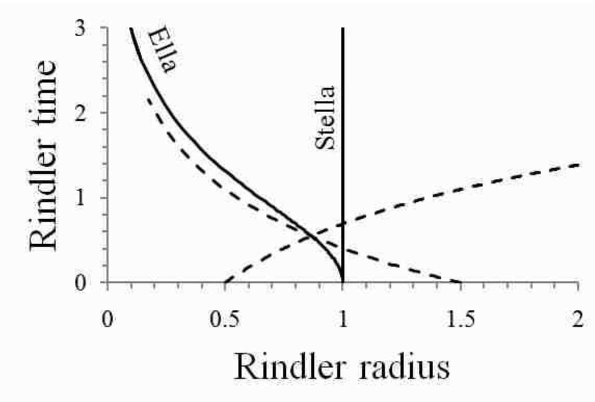
For the freely propagating astronaut Ella we find

$$c\theta = \tanh^{-1} ct \qquad \rho = \sqrt{1 - c^2t^2} \quad (17.4)$$

and hence

$$\rho = \frac{1}{\cosh c\theta}. \quad (17.5)$$

This world-line is illustrated to the right.

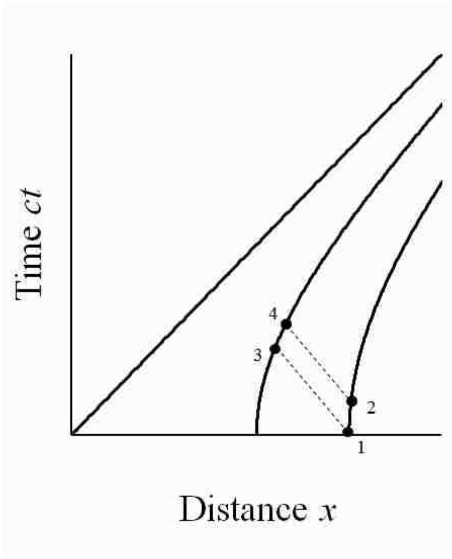


Initially Stella sees Ella appear to fall away from her parabolically (i.e. change in ρ scales with θ^2). As her friend falls towards $\rho = 0$, Stella notices that Ella appears to be ageing slowly. In fact her wrist-watch reveals that time itself appears to be passing ever more slowly for Ella, from Stella's perspective. Even the photons conveying Ella's image appear to have been forged using this bogus dilated time, for Stella perceives her friend to have an increasingly red appearance. Thus Ella's image grows increasingly dim (as fewer photons per second of Stella's time are carrying her image) and increasingly red-shifted. To Stella, Ella appears to forever hang over the spatial origin ($\rho = 0$), always getting closer but never arriving. Of course the inertial S frame observer will explain the red-shift as a natural consequence of the relative motion between source and detector, i.e. it is the Doppler effect we saw in §10.3. And the S frame observer will not be surprised to see that Ella's world-line above is not straight from Stella's perspective - it is simply the effect of using the strange curvy co-ordinate system from the previous page.

17.4 Propagation of Light Through Rindler Space

Rather than ejecting Ella from her spaceship, suppose Stella emits a photon at $t = 0$. And suppose the photon is emitted in the y direction. Having no velocity in the x direction, the projection of this photon's world-line onto the x-ct plane would be indistinguishable from Ella's in the diagram above, but it would be inclined at 45° to the ct axis in the y-ct plane (since $y = ct$). Using Eq. 17.2 we find $\rho = \sqrt{1 - y^2}$ and hence for small times the change in the Rindler radius of the photon is $-\frac{1}{2}y^2 a_0/c^2$ when it travels a distance y perpendicular to Stella's acceleration: its path through space appears to be curved! (Compare to the diagram in §16.5.)

The world-lines of photons fired at Stella along the x direction from $\rho = 0.5$ and $\rho = 1.5$ are shown as dashed lines in the Rindler diagram above. It is clear that light follows curved trajectories in *space-time* as well as through space, and hence the speed of light is not constant in Rindler space. Note however that the speed of light measured *locally*, i.e. as it passes by Stella's location, is c.



Similarly we can determine the Doppler shifts discussed qualitatively in §16.4. Events 1 and 2 in the S frame diagram to the right correspond to the emission of successive crests of a light wave leaving an observer stationary in Stella's rest frame while 3 and 4 are these crests reaching Stella. Eq. 17.2 allows us figure out the proper time elapsed between events 1 and 2 in the emitting observer's frame and the proper time that elapses for Stella between events 3 and 4. In this way, the first order treatment from §16.4 is verified.

The treatment of acceleration in this chapter provides a more enlightened justification for the results of §16.4-16.5. In the next chapter we will see it allows us to see considerably further into "General Relativty".