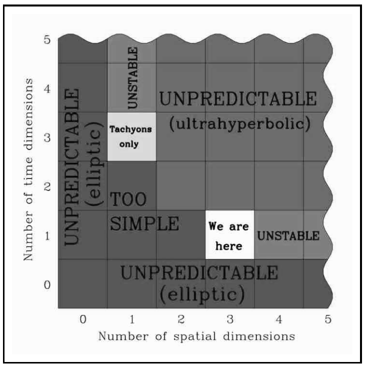



Tegmark *et al.*

# Chapter 18

# General Relativity: Einstein's Theory of Gravitation

## 18.1 The Metric Tensor

The square of the distance between two nearby points in conventional 3D space, or "line element", can be written:

$$(dx)^2 + (dy)^2 + (dz)^2 = \begin{pmatrix} dx & dy & dz \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \tag{18.1}$$

$$= \begin{pmatrix} dr & d\theta & d\phi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix} \tag{18.2}$$

The matrix inserted between the row and column vectors is known as the *metric tensor*, often denoted $g_{\mu\nu}$. In 3D Euclidean space the metric tensor for Cartesian co-ordinates is the identity matrix. If we choose to move to spherical polar co-ordinates $(r, \theta, \phi)$ we know that $(dr)^2 + (d\theta)^2 + (d\phi)^2$ does not give a correct measure of distance - rather we use the metric in the second expression above.

In §16.6 we noted that in spaces with intrinsic curvature the familiar laws of Euclidean geometry, by definition, fail. In these cases a non-trivial metric is a necessity. The metric tensor is the vital machinery of a branch of mathematics known as "differential geometry", providing the means to cope with the geometric distortion of curved vector spaces. In general the elements of $g_{\mu\nu}$ vary with position, and off diagonal elements are non-zero when the co-ordinate axes are not orthogonal. Technically we should speak of a scalar product as a component by component product of a contravariant vector and its covariant dual, e.g. $ds^2 = dx_\mu \, dx^\nu$, with the metric tensor converting between them, e.g. $dx_\nu = g_{\mu\nu} dx^\mu$, and hence $ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu$. Here we will pursue a non-rigorous qualitative discussion, almost entirely free of tensors, subscripts and superscripts.

In §11 we stated that for Minkowski space (i.e. within Special Relativity) the equivalent quantity to a distance element is "the interval", $ds$ defined by:

$$(ds)^2 = -c^2\,(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2 = \begin{pmatrix} cdt & dx & dy & dz \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\,dt \\ dx \\ dy \\ dz \end{pmatrix}. \tag{18.3}$$

In the previous Chapter we saw that reality for an observer accelerating through "flat" Minkowski space appears to have been painted onto a curved canvass. In that case, (since $x = \rho \cosh c\theta$ and $ct = \rho \sinh c\theta$) the space-time line element takes the form:

$$(ds)^2 = -\rho^2\,c^2\,(d\theta)^2 + (d\rho)^2 + (dy)^2 + (dz)^2 = -c^2\,2\Re\,(d\theta)^2 + \frac{(d\Re)^2}{2\Re} + (dy)^2 + (dz)^2, \tag{18.4}$$

where we introduced a new variable $\Re = \rho^2/2$ to obtain the final equality. Use of an accelerated frame has the effect of curving the co-ordinate system.[1] By the Equivalence Principle (§16) gravity also implies space-time curvature.

## 18.2 Einstein's Field Equations

Einstein proposed that the curvature of space-time is related to the distribution of mass by the expression:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^2} T_{\mu\nu}. \tag{18.5}$$

$R_{\mu\nu}$ is the "Ricci curvature tensor" and is determined by the derivatives of the metric tensor, $g_{\mu\nu}$, with respect to the position 4-vector. $R$ is the local curvature and is given by $g^{\mu\nu} R_{\mu\nu}$. $G$ is Newton's gravitational constant and $T_{\mu\nu}$ is the "matter tensor" which specifies the distribution of 4-momentum. We are avoiding any discussion of tensors in this course so don't worry too much about this expression, but you should be able to appreciate that the left hand side of the equation involves geometric quantities related to curvature, while the right hand side describes the distribution of gravitating matter. This is essentially an up-market version of Eq. 16.7.

Most of the time we are concerned with gravitational fields arising from spherically symmetric bodies, such as stars and planets. Using spherical polar coordinates, the Minkowski metric is:

$$(ds)^2 = -(c\,dt)^2 + (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2\theta (d\phi)^2. \tag{18.6}$$

For the spherically symmetric gravitational field due to a body of mass $M$, the distorted metric must have the form:

$$(ds)^2 = -e(r)\,(c\,dt)^2 + f(r)\,(dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2\theta\,(d\phi)^2. \tag{18.7}$$

In 1916, Schwarzschild found the following solution to Einstein's field equation *outside* a spherical mass:

$$e(r) = 1 + 2\Phi/c^2 \qquad f(r) = \frac{1}{1 + 2\Phi/c^2} \tag{18.8}$$

where $\Phi(r)$ is the gravitational potential, $-GM/r$. We will not discuss the derivation of the "Schwarzschild metric", but consideration of its implications is illuminating.

## 18.3 Properties of the Schwarzschild Solution

**1. Static**
The $e$ and $f$ functions do not depend on $t$. The Schwarzschild solution is "static".

**2. Singularity**
At $r = 0$ there is a singularity as $\Phi \to -\infty$ and hence $e(r) \to -\infty$.

**3. Distant Limit**
Since $e(\infty) = f(\infty) = 1$ we recover flat Minkowski space-time far from the mass.

**4. Curvature**
The factor $2\Phi/c^2$ is a measure of the curvature at radial coordinate $r$. Some typical values are:

$$\begin{aligned} \text{At the surface of the Sun:} \quad 2\Phi/c^2 &\simeq 4 \times 10^{-6} \\ \text{At the surface of the Earth:} \quad 2\Phi/c^2 &\simeq 1.4 \times 10^{-8} \end{aligned}$$

Since we reside in an $\Phi/c^2 \ll 1$ environment, most of us do not have an instinct for space-time curvature.

**5. Schwarzschild radius**
Inspection of Eq. 18.8 suggests that something special happens as $r$ approaches $2GM/c^2$, known as the "Schwarzschild radius" and denoted $r_S$. For a body with the mass of the Sun, $r_S = 2.95$ km, whilst for the Earth it is 8.86 mm, much smaller than their physical sizes. Since the Schwarzschild solution is only valid

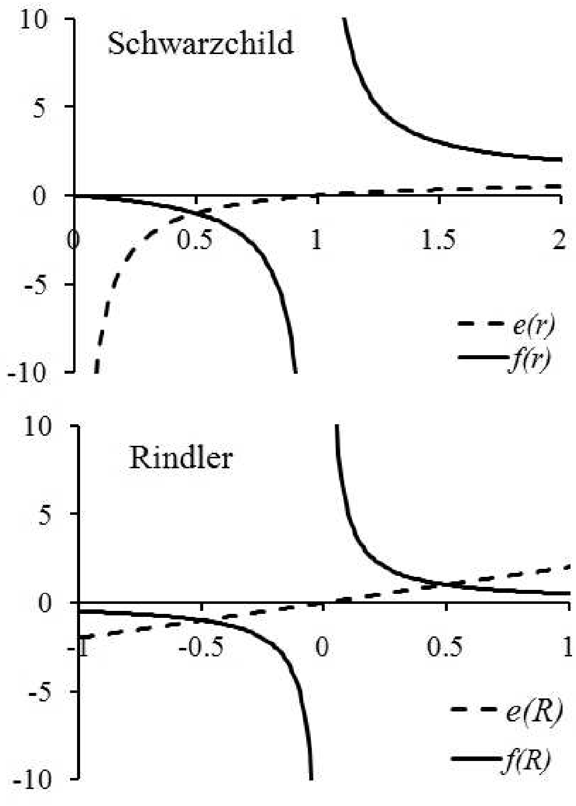
[1] Note that the accelerating observer's "Cartesian" space-time co-ordinate system, i.e. $c\theta, \rho, y, z$, is curved - this is not just a matter of choosing a weird change of variables.

*outside* an object, it is clear that we must find an unimaginably dense body if we are to experience life near the Schwarzschild radius. But we can also employ the imagination.

Consider approaching $r_S$ from $r \gtrsim r_S$. We noted in §11.4 that for a space-like interval $(ds)^2$ measures "proper" distance, $(d\ell)^2$, and for a timelike interval it measures $-c^2(d\tau)^2$. Approaching the Schwarzschild radius we find $e \to 0$, and so any finite proper time will correspond to a divergent period of co-ordinate time, $dt \to \infty$. This is highly reminiscent of our discussion of Ella falling from Stella's accelerating frame in §17.3. Let us explore the parallel between Schwarzschild and Rindler space in more detail.

## 18.4 Schwarzschild and Rindler

In §18.1 we presented the Rindler metric using $\Re$ and $\theta$ rather than $\rho$ and $\theta$. It seems that the second equality of Eq. 18.4 is rather similar in form to the Schwarzschild metric in Eq. 18.8, as illustrated in the plots below. However we should not expect Rindler space to provide a perfect analogue for the space-time curvature around a gravitating mass since the latter is spherically symmetric while the former describes uniform linear acceleration. Nevertheless, the similarity between the $e$ and $f$ functions in the two cases suggests that the Schwarzschild radius plays the role of the Rindler event horizon.




### 18.4.1 Near-Horizon Approximation

We start with the Schwarzschild line element:

$$(ds)^2 = -(c\,dt)^2\,\frac{r - r_S}{r} + (dr)^2\,\frac{r}{r - r_S} + r^2(d\Omega)^2 \tag{18.9}$$

with the angular parts from Eq. 18.7 written in more compact form. In the near horizon approximation, $r \approx r_S$, we obtain:

$$(ds)^2 = -(c\,dt)^2\,\frac{r - r_S}{r_S} + (dr)^2\,\frac{r_S}{r - r_S} + r^2(d\Omega)^2. \tag{18.10}$$

Considering now a proper radial displacement, i.e. $dt = 0$ and $d\theta = d\phi = 0$ (or $d\Omega = 0$), this equation becomes

$$d\ell = dr\sqrt{\frac{r_S}{r - r_S}}. \tag{18.11}$$

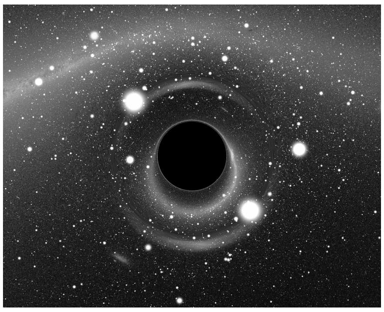
We can integrate this to set $\ell$ equal to the proper radial distance from the Schwarzschild horizon:

$$\ell = \int_{r_S}^{r} dr'\,\sqrt{\frac{r_S}{r' - r_S}} = 2\sqrt{r_S}\sqrt{r - r_S}. \tag{18.12}$$

When we are close to some point on the Schwarzschild sphere, $r = r_S$, the unit vectors $\underline{\hat{e}}_\theta$ and $\underline{\hat{e}}_\phi$ can be taken as the Cartesian $\underline{\hat{y}}$ and $\underline{\hat{z}}$ vectors, and it follows that Eq. 18.10 can be written

$$(ds)^2 = -(c\,dt)^2\,\frac{\ell^2}{4r_S} + (d\ell)^2 + (dy)^2 + (dz)^2. \tag{18.13}$$

Thus the Schwarzschild solution for a spherical gravitating mass leads, in the near horizon region, to the same curved space-time geometry as seen in Stella's uniformly accelerating frame in the previous Chapter.



### 18.4.2 Black holes

The energy output of stars is sustained by nuclear fusion. When the nuclear fuel starts to run out, the star cools and its internal pressure falls. Internal gravitational attraction tends to induce collapse. If the star's radius reaches the Schwarzschild radius, $r_S$, its density is so great that the resulting curvature of space-time prevents anything escaping from the surface of the star. Even light does not escape. The Schwarzschild radius is an *event horizon* for an observer outside, analagous to the Rindler horizon of the previous Chapter. The inner region is therefore known as a *black hole*.

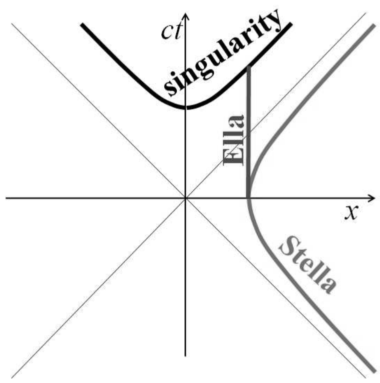
Not all stars suffer the fate of gravitational collapse to a black hole; a critical mass is necessary. For stars with mass below the "Chandrasekhar-Landau limit" the Heisenberg Uncertainty Principle provides a balancing force to counter gravitational collapse. However the zero point motion required by quantum mechanics cannot give rise to particle velocities in excess of $c$, and so there is a limit to the strength of the uncertainty effect. For star masses greater than the Chandrasekhar-Landau limit gravitation wins.

### 18.4.3 Ella's Fate

Let us consider briefly the gravitational anaologue of the Ella scenario from §17.3. There we saw Ella ejected from Stella's accelerating spaceship and noted that in the inertial $S$ frame Ella's motion would be quite uneventful - she would obey Newton's first law and coast along at constant speed in a straight line. But from Stella's perspective Ella hangs over the event horizon for all eternity. The Equivalence Principle, bolstered by our examination of the Near Horizon Approximation of the Schwarzschild metric, would indicate that Ella appears to suffer the same indignity when falling towards the event horizon of a black hole.

It appears from Eq. 18.9 that we have a major mathematical problem at $r = r_S$. On passing $r_S$ the signs of $e$ and $f$ change, and since the $(dt)^2$ and $(dx)^2$ terms enter $(ds)^2$ with opposite sign, it is often said that time and space switch roles. Meanwhile Ella experiences no particular sensation as she crosses the Schwarzschild horizon. The infinities at $r_S$ are not physical singularities but rather of purely mathematical origin, and they can be removed by yet another change of co-ordinates.

There are some problems for Ella though. Firstly she will sense, at some point, so-called "tidal forces". This is the distortion (and eventual ripping apart) arising when there is significant variation in gravitational force from one part of a body to another. Then there is a fearsome (physical) singularity at $r = 0$. According to Eq. 18.12, crossing the Schwarzschild horizon renders $\ell^2$ increasingly negative, reaching $-4r_S^2$ at the $r = 0$ singularity. Appealing to the direct parallel discovered in §18.4.1 between $\ell$ for the Schwarzschild metric and $\rho$ for the Rindler metric, we recall from §17.2 that the gridlines of constant $\rho$ consisted of hyperbolae in the $x$-$ct$ space-time of frame $S$. There we considered only +ve values of $\rho^2$. A -ve $\rho^2$ would also correspond to hyperbolae, but in the forward light cone, as illustrated in the image to the right. Thus our earlier "Ella diagram" was deficient (as a model of a black hole) in one important respect - time ends when Ella's world-line hits the hyperbola of the singularity, as it inevitably will.




## 18.5 Motion in Curved Space-time

Gravity appears to distort space-time according to Einstein's field equations. Now we specify how matter moves within this curved space-time. World-lines of free particles are said to be *geodesics*, but we used that term in §3.2.1 to mean the shortest distance between two points. Let's take a moment to establish the connection.

Consider how a particle near a clock on the Earth's surface should move if we impose the boundary condition that it must return to the starting point when 10 seconds have elapsed on the clock. We could consider this classical problem using the version of Hamilton's *Principle of Least Action* seen in §4. The particle must minimize the integral of its Lagrangian, $T - V$, so moving to higher altitude (where $V$ is very negative) is a good strategy. But it should be careful not to go too far or it will need to travel very quickly in order to get home in time, and high kinetic energy is a bad idea. It turns out that the optimum balance of going up high and not needing to go too fast is the parabola.

We have seen that the generalization of the 3D distance is the *interval*, $s$, in 4D space-time. For any particular journey minimizing $s$ means maximizing the *proper time* for the journey. We showed in Problem Sheet 8 that $\tau$ is maximized for a free particle by a straight world-line in Minkowski space. Taking the low speed approximation we see that maximizing $\tau$ appears to correspond to minimizing $\frac{1}{2}mu^2 - mc^2$. Could it be that $-mc^2\sqrt{1 - u^2/c^2}$ is the appropriate relativistic form for $\mathcal{L}$, the Lagrangian (for a free particle at least), and that Hamilton's Principle of Least Action can be re-stated as the *Principle of Most Proper Time*? Yes!

*A particle moves to maximize its proper time.*

So reconsider our particle. He wishes to reach high altitude so that it can be in a region where time for him will pass quickly, helping to maximize $\tau$. But again the particle must take care not to go too high because that will require high speed (in order to satisfy the boundary conditions) and time will pass slowly for him, according to the stationary clock. In the low speed and weak field limits we see that the SR time dilation effect scales with $1/\gamma \rightarrow 1 - v^2/2c^2$ and the gravitational time dilation effect scales with $gh/c^2$, where $g$ is the magnitude of the acceleration and $h$ the distance moved. Thus maximizing $\tau$ means maximizing $1 - v^2/2c^2 + gh/c^2$, or minimizing $mv^2/2 - mgh$. So the proper time version of Hamilton's Principle implies also the old version from §4 but also works for the high speed strong field case too.

We can now state Einstein's theory of gravitation thus:

- Matter induces curvature to space-time according to Einstein's field equations (Eq. 18.5).
- Objects move in gravitational fields so as to maximise their proper time ("Einstein's Equation of Motion").

The gravitational interaction of two masses is sometimes likened to the movement of two bowling balls on a trampoline. Each influences the other by curving the trampoline in its locality.

## 18.6 Cosmology

Einstein's theory of General Relativity was to revolutionise cosmology - our understanding of the universe as a whole. While space and time are inert concepts in classical physics and even in Special Relativity, they become players in the game according to GR. Since space-time can no longer be considered as some passive, eternal framework, new questions were asked about it and of the universe. Since we are now contemplating the nature of existence, this is very interesting, but potentially quite explosive as our deductions impinge on philosophy and theology.

### The Dynamic Universe

The idea that space-time is distorted by the gravitational effect of mass forces us to see the universe as a dynamic entity. An obscure Russian called Friedmann (and not Einstein) was the first to embrace this by suggesting that the extent of the universe may not be fixed. In 1922 Friedmann showed that if the universe looks the same in all directions and from all points then it *cannot be static* (i.e. it must be either expanding or contracting). Of course matter is not distributed uniformly throughout the universe - it is clumped up into planets and stars and galaxies etc. But it turns out that on a coarse scale Friedmann's first assumption is rather good. Given that our solar system does not seem in any way special, Friedmann's second assumption seems intuitively valid. But was his deduction correct?

In the 1920s astonomers (notably Hubble) started to make observations of distant galaxies. They found that the vast majority of galaxies are moving in a quite systematic way - they move away from us (as determined by the Doppler shift of their stellar optical spectra) at a rate proportional to their distance from us. The universe is expanding.

### The Fate of the Universe

Will the internal gravitational attraction within the universe eventually bring its expansion to a halt (or even reverse it)? To address this question we need to know (i) the current rate of expansion, and (ii) the amount of matter in the universe. Astronomers can provide a reasonable estimate of (i), but (ii) is harder. The problem is that it is not enough to only count the matter that we can see. The movement of many stars and galaxies cannot be explained by the visible matter alone - there must be gravitating matter which we cannot see. Cosmologists' best guesses at the total amount of *dark matter* in the universe suggest that there is insufficient mass to halt the current expansion. Indeed it appears that the rate of expansion of the universe is actually increasing, as if something were driving it. This "something" is often referred to as "dark energy". Since we have little idea of what dark matter and dark energy actually are, we cannot be wholly confident that these ideas are correct. A critic might say that these are just technical names for our ignorance, but this is how science works.

### The Birth of the Universe

If our universe is expanding, then in the past it was hotter and more dense than it is today. Indeed Friedmann's model implies, whatever the fate of the universe, it "began" with infinite density at a single point, i.e. a singularity. The laws of physics would breakdown under these circumstances, rendering the question "what happened before the singularity?" irrelevant (since such events could not influence the post-singularity universe) if not without meaning altogether. In the Friedmann model the universe effectively "begins" with a sudden expansion known as the "big bang".

Though the big bang model is widely accepted today it was not always so, and there appears to be something unsatisfactory about a scientific theory which requires a singularity at which the theory is itself invalid. The question of whether GR required (or even allowed) singularities, or whether this was an artefact of Friedmann's approximations, was addressed about thirty years ago by Penrose and Hawking. Penrose showed that if a star, exhausted of the nuclear fuel that sustains it, undergoes gravitational collapse (i.e. its own enormous mass makes it fall in on itself), then GR dictates that it will inevitably collapse to a singularity. Using a time-reversed version of the Penrose theory, Hawking argued that GR requires the universe to have begun from a singularity. In closing this section we note a problem. As we go backwards in time towards the big bang, we inevitably reach a point where a quantum mechanical form of GR is required. We don't really know too much about the implications of this point - first we require a "grand unified theory".

We might hope that such a theory could predict various "constants" used by (but not derived in) current theories in physics (e.g. $k_B, c, G, m_e, \hbar, m_p$ etc.) and the number of temporal and spatial dimensions. It appears that the universe is suspiciously finely-tuned for our existence.

We don't yet have the answers, but it's fun to wonder.