Dynamics and Relativity
Solutions 10:

1. a) For an elastic collision, the total kinetic energy T is conserved, where T is the total energy minus the sum of the rest energies. We showed in §13.6 that E is conserved (in the absence of external forces, since 3-momentum must be conserved). It follows that the total rest energy must be conserved, and hence mass is conserved.

   b) In the classical regime, energy conservation implies

$$\frac{p^{*2}}{2m_1} + \frac{p^{*2}}{2m_2} = \frac{q^{*2}}{2m_1} + \frac{q^{*2}}{2m_2}.$$

   With no special assumptions about the masses m_1 and m_2, it follows that $p^{*2} = q^{*2}$.

   In the relativistic case, the kinetic energy of a particle is

$$T = E - mc^2 = \sqrt{m^2c^4 + p^2c^2} - mc^2.$$

   For two particles with initial and final momenta $\pm p^*$, $\pm q^*$ then conservation of T means

$$\sqrt{m_1^2c^4 + p^{*2}c^2} - m_1c^2 + \sqrt{m_2^2c^4 + p^{*2}c^2} - m_2c^2 = \sqrt{m_1^2c^4 + q^{*2}c^2} - m_1c^2 + \sqrt{m_2^2c^4 + q^{*2}c^2} - m_2c^2.$$

   As before, this can only be true if $p^* = q^*$.

   If $m_1 = m_2 = m$, and given $p^* = q^*$, then we find $E_1^* = E_2^* = E_1'^* = E_2'^* = \sqrt{m^2c^4 + p^{*2}c^2}$, i.e. the energy of each particle is $\sqrt{m^2c^4 + p^{*2}c^2}$ before and after the collision.

2. In the LAB frame, the 4-momenta of incident and target particles before the collision are $\underline{\underline{p}}_1 = [E_1/c, p_1, 0, 0]$ and $\underline{\underline{p}}_2 = [mc, 0, 0, 0]$. After the collision the scattered and recoiling particles have 4-momenta $\underline{\underline{q}}_1 = [E_1'/c, q_1\cos\theta, q_1\sin\theta, 0]$ and $\underline{\underline{q}}_2 = [E_2'/c, q_2\cos\psi, -q_2\sin\psi, 0]$. It is easier to work in the CM frame in which the initial 4-momenta are $[E^*/c, p^*, 0, 0]$ and $[E^*/c, -p^*, 0, 0]$, and the final momenta are $[E^*/c, p^*\cos\theta^*, p^*\sin\theta^*, 0]$ and
$[E^*/c, -p^*\cos\theta^*, -p^*\sin\theta^*, 0]$. In writing down these expressions we have used results from Q1 (i.e. $p^* = q^*$ and $E_1^* = E_2^* = E_1'^* = E_2'^*$) which in turn relied upon 4-momentum conservation.

   $\tan\theta$ is equal to q_1^y/q_1^x, the ratio of the y and x components of the LAB frame 4-momentum of the scattered particle. Assuming the CM frame travels with speed v_{cm} in the LAB frame, we can use the inverse Lorentz transform to write:

$$\tan\theta = \frac{q_1^y}{q_1^x} = \frac{p^*\sin\theta^*}{\gamma(v_{cm})(p^*\cos\theta^* + v_{cm}E^*/c^2)}.$$

   Similarly we obtain

$$\tan\psi = \frac{-q_2^y}{q_2^x} = \frac{p^*\sin\theta^*}{\gamma(v_{cm})(-p^*\cos\theta^* + v_{cm}E^*/c^2)}.$$

   To develop these expressions further, we must eliminate v_{cm} (by exploiting knowledge of the link between the CM and LAB frames). To do this we consider the inverse L.T. for the target particle:

$$0 = \gamma(v_{cm})(-p^* + v_{cm}E^*/c^2) \qquad \Rightarrow \qquad v_{cm}E^*/c^2 = p^*.$$

Using this expression we obtain:

$$\tan\theta = \frac{\sin\theta^*}{\gamma(v_{cm})(\cos\theta^* + 1)} \qquad \tan\psi = \frac{\sin\theta^*}{\gamma(v_{cm})(-\cos\theta^* + 1)}.$$

The desired result

$$\tan\theta\ \tan\psi = \frac{1}{\gamma^2(v_{cm})}$$

follows immediately.

In the non-relativistic limit, $\gamma \to 1$ which means $\theta + \psi \to \pi/2$. In the relativistic case, the sum of these angles is less than 1, giving enhanced scattering in the forward direction.

3. The CM frame 4-momenta before the collision are $[E^*/c, p^*, 0, 0]$ and $[E^*/c, -p^*, 0, 0]$ while after the collision we have $[E^*/c, p^*\cos\theta^*, p^*\sin\theta^*, 0]$ and $[E^*/c, -p^*\cos\theta^*, -p^*\sin\theta^*, 0]$, where again we have used the results from Q1b.

   We require the electron directions in the LAB frame so we must make an inverse L.T. for the post-collision 3-momenta. For electron 1 (the incident particle):

$$q_{x1} = \gamma(v_{cm})(p^*\cos\theta^* + v_{cm}E^*/c^2), \qquad q_{y1} = p^*\sin\theta^*.$$

   Since we know E^* and mc^2, we can deduce $\gamma(v_{cm}) = 2$ from the Lorentz transformation which gives the pre-collision energy of the target particle in the CM frame:

$$\frac{E^*}{c} = \gamma(v_{cm})\left(mc - \frac{v_{cm}}{c}.0\right) \qquad \Rightarrow \qquad \gamma(v_{cm}) = 2, \quad v_{cm}/c = \sqrt{3}/2.$$

   Similarly, the CM 3-momentum of the target particle can be written

$$-p^* = \gamma(v_{cm})\left(0 - \frac{v_{cm}}{c}E^*\right) \qquad \Rightarrow \qquad p^*c = \sqrt{3}/2 \text{ MeV}.$$

   We now know all the quantities needed to obtain θ:

$$\tan\theta = \frac{q_{y1}}{q_{x1}} = \frac{p^*\sin\theta^*}{\gamma(v_{cm})(p^*\cos\theta^* + v_{cm}E^*/c^2)}.$$

   We obtain $\theta = 12°$.

   We follow the same approach for particle 2:

$$q_{x2} = \gamma(v_{cm})(-p^*\cos\theta^* + v_{cm}E^*/c^2), \qquad q_{y2} = -p^*\sin\theta^*$$

$$\tan\psi = q_{y2}/q_{x2} \qquad \Rightarrow \qquad \psi = 50°.$$

   To obtain the LAB frame particle post-collision energies E_1' and E_2', again we use the inverse L.T.:

$$E_1' = \gamma(v_{cm})(E^* + v_{cm}p^*\cos\theta^*) = 2\left(1 + \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}\right) = 3.06\,\text{MeV}$$

$$E_2' = \gamma(v_{cm})(E^* - v_{cm}p^*\cos\theta^*) = 2\left(1 - \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}\right) = 0.94\,\text{MeV}.$$

4. The 4-momenta beforehand are $[\gamma(u_1)m_1c, \gamma(u_1)m_1u_1, 0, 0]$, $[\gamma(u_2)m_2c, \gamma(u_2)m_2u_2, 0, 0]$, while that after coalescence we write as $[\gamma(u_3)m_3c, \gamma(u_3)m_3u_3, 0, 0]$ (since the particles are colinear). Conservation of 4-momentum requires:

$$\gamma(u_1)m_1c + \gamma(u_2)m_2c = \gamma(u_3)m_3c \tag{1}$$
$$\gamma(u_1)m_1u_1 + \gamma(u_2)m_2u_2 = \gamma(u_3)m_3u_3. \tag{2}$$

Dividing (2) by (1) gives

$$u_3 = \frac{\gamma(u_1)m_1u_1 + \gamma(u_2)m_2u_2}{\gamma(u_1)m_1 + \gamma(u_2)m_2}$$

as required.

The next expression we are aiming for contains m_3^2 but not u_3. Note that u_3 enters Eq. 2 explicitly and but also sneaks into Eq. 1 and 2 in $\gamma(u_3)$. We can get rid of it by squaring the two equations and then adding, since $\gamma^2(c^2 - u^2) = c^2$. In this way we obtain:

$$[\gamma(u_1)m_1c + \gamma(u_2)m_2c]^2 - [\gamma(u_1)m_1u_1 + \gamma(u_2)m_2u_2]^2 = m_3^2c^2.$$

After simplifying the algebra we get the necessary result.

An alternative (and much easier) method is suggested by the presence of squared masses in the desired expression. Evaluating m_{12}^2 using the total 4-momenta before and after the collision gives the desired result directly.

5. The "invariant mass" m_{12} is a useful concept here. This is defined by

$$m_{12}^2c^2 = -\underline{\underline{P}} \cdot \underline{\underline{P}}$$

where $\underline{\underline{P}} = \underline{\underline{p}}_1 + \underline{\underline{p}}_2$ is the total momentum before the collision. The beauty of this is (i) $\underline{\underline{P}}$ is the same before and after the collision, and (ii) since scalar products are invariant m_{12} can be evaluated in any inertial frame.

For proton-proton collisions in the LAB frame,

$$m_{12}^2c^2 = -(\underline{\underline{p}}_1 + \underline{\underline{p}}_2) \cdot (\underline{\underline{p}}_1 + \underline{\underline{p}}_2) = m_p^2c^2 + m_p^2c^2 + 2m_pE_1$$

where $\underline{\underline{p}}_1$ and $\underline{\underline{p}}_2$ are the 4-momenta of the two protons, and proton 2 is assumed to be at rest. We can evaluate m_{12}^2 after the collision in the CM frame. In this frame we know that $m_{12}c^2$ equals the total energy. The minimum possible value m_{12} can take is when the post-collision particles are at rest (i.e. with no kinetic energy) in the CM frame. Thus:

$$(m_{12})_{min} = m_p + m_p + m_\pi$$

and the corresponding value of E_1 is obtained by equating the previous two expressions:

$$(m_p + m_p + m_\pi)^2c^2 = m_p^2c^2 + m_p^2c^2 + 2m_p(E_1)_{min}.$$

Finally we obtain

$$(E_1)_{min} = \frac{(m_p + m_p + m_\pi)^2c^4 - (m_p^2 + m_p^2)c^4}{2m_2c^2}.$$

Using the given masses, we get:
$(E_1)_{min} = 1218$ MeV for (a), and $(E_1)_{min} = 6580$ MeV for (b).

6. Using the given numbers we get:

System	m (in MeV/c^2)	L	λ_c/L
e in atom	0.5	10^{-10}	0.02
p in nucleus	935	10^{-14}	0.12
quark in proton	2.4	10^{-15}	500

Thus our qualitative consideration of electron KE in an atom suggests energies way below the rest energy, removing pair production from the picture. Thus we are justified in treating the electron in the H atom as a one body problem in the Schrödinger (non-relativistic) or Dirac (relativistic) equations. At the other end of the scale our calculations suggest that quark-quark interactions within nucleons are exceedingly complex.