The tippe top revisited

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The tippe top displays the unexpected behavior of inverting itself when it is spun upon a flat surface. In the process its center of mass rises. Several explanations of this behavior have been offered in the literature. Here, in the light of a historical perspective, physical arguments are presented which support the contention that the influence of sliding friction is the key to the understanding of the top's behavior. Then, a rigorous analysis of the top's mechanics is offered, together with computer-generated solutions of the equations of motion.

The classical dime-store model tippe top (Fig. 1) consists of a section of a sphere upon whose planar surface is mounted a short rod. The sportsman spinning this device will note that this perverse top refuses to sit on its rounded head; it proceeds to flip over and rotate on its elongated stem. In the process of inverting itself the center of mass of the object is raised. Also, the direction of rotation reverses with respect to the body axes as the top turns over. One may experimentally verify that this inversion phenomenon is a property of many rounded bodies in which the center of mass is eccentric. Indeed, the author first became intrigued with the analysis of this unexpected behavior while idly spinning his college ring; when this object is spun with the heavy stone bottommost, the ring promptly rights itself hefting the stone superiorly.

John Perry in his amusing book, *Spinning Tops and Gyroscopic Motion,*[1] claims that an attempt at a resolution of the problem was made by Archibald Smith in the *Cambridge Mathematical Journal* during the 19th century. Perry reveals that Sir William Thomson (1824–1907) and Professor Blackburn had attacked the problem while they were supposed to have been studying for the "great Cambridge mathematical examination." (Instead, they spent their time spinning smooth stones which they picked up on the beach—a testimony to the tippe top's insidious nature.) Perry also reveals that a mathematical approach was attempted by Jellet in his *Treatise on the Theory of Friction* published in 1872. More recently, Professor E. M. Purcell[2] reveals that Niels Bohr was also interested in this problem.

The first essentially accurate modern explanations of the top's behavior came to light in a series of articles by Braams,[3] Hugenholtz,[4] and Pliskin[5] during the early 1950's—when the tippe top craze amongst physicists was

Frontispiece. Wolfgang Pauli and Niels Bohr fascinated by a spinning top. (Permission to reprint from AIP Niels Bohr Library.)

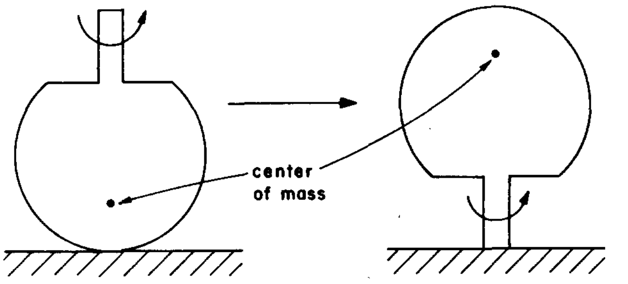

Fig. 1. Tippe top inverting.

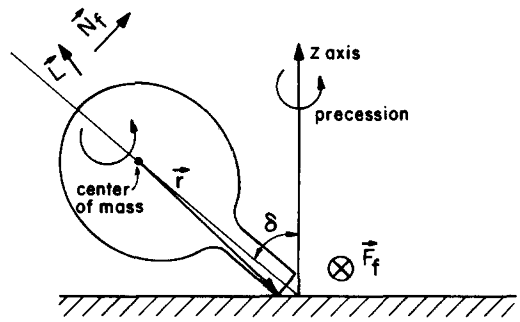

Fig. 2. Conventional top.

at its height. However, their treatment is limited by the nature of the approximations needed to perform the calculations analytically.

The tippe top's motion constitutes the sort of phenomenon abundant in physics, for which a simple physical analysis reveals the underlying principles, yet for which a detailed and rigorous solution (which may require the use of computing machines) is necessary to confirm the analysis.

Pliskin, Braams, and Hugenholtz contended that the frictional interaction of the top with the table surface plays a decisive role in the inversion of the top while Synge,[6] another investigator, contended that friction was not important but rather that an asymmetrical distribution of mass about the spin axis was the crucial ingredient.

Del Campo[7] demonstrated from very general considerations that friction must be important. Here in brief is his argument.

Let us define the z axis to be perpendicular to the table surface. The acute observer will note that during inversion the z component of angular momentum remains dominant. More specifically, both before and after inversion the angular momentum points almost entirely along the positive z axis. Thus, the direction of rotation of the top with respect to the coordinates fixed in its body is reversed. During inversion the center of mass of the top is elevated; it follows that the rotational kinetic energy decreases during inversion in order to provide the potential energy involved in this raising of the center of mass. This necessarily implies that the total angular velocity and the total angular momentum decrease during the inversion process. However, a reduction in angular momentum requires the action of a torque. If no frictional forces are present, the only external forces acting on the top are gravity and the normal force exerted by the table at its point of contact with the top. These forces point along the z axis and cannot produce torques along the z axis; hence, they cannot be responsible for the change in the z component of the angular momentum. Therefore, a correct explanation of the top's behavior must involve frictional forces.

Before proceeding with a discussion of the mechanics of the tippe top, it is instructive to briefly examine the influence of friction on the rising of a conventional top. Consider a top, consisting of a symmetrical mass, mounted on an axle of finite radius, set spinning about its axis of symmetry on a frictionless table at an angle δ to the vertical (Fig. 2). The angular momentum $\mathbf{L}$ is not quite parallel to the axis of the top, since there is a precessional motion about the top's center of mass in addition to the more rapid rotatory motion (spin) of the top about its own axis of symmetry. Now, introduction of a small coefficient of friction induces a frictional force which opposes the motion of the edge of the stem with respect to the table surface. At the point of contact the instantaneous velocity of the edge of the stem is primarily a result of the spin of the top about its axis of symmetry. Hence, the frictional force as pictured in Fig. 2 points into the paper. This force produces a torque $\mathbf{N}_f$ about the center of mass:

$$\mathbf{N}_f = \mathbf{r} \times \mathbf{F}_f. \tag{1}$$

$\mathbf{N}_f$ is perpendicular to the vector $\mathbf{r}$, and will tend to align the angular momentum $\mathbf{L}$ with the vertical. The orientation of the top follows that of $\mathbf{L}$, and the top rights itself.

This sort of an analysis, however, will not suffice to account for the behavior of the tippe top. For one thing, this analysis depends on the fact that $\mathbf{L}$ points predominantly along the symmetry axis of the top, whereas the angular momentum of the tippe top points predominantly along the positive z axis during the entire inversion process. Moreover, a correct analysis of the tippe top must take into account the change in the vector $\mathbf{r}$ as a function of the angle of inclination. To proceed with the analysis of the tippe top's motion, we introduce the symbols listed in Table I in conjunction with Figs. 3 and 4. The only restriction relating to the mass distribution inside the top is that it be symmetric about the shaft and that the center of mass be epicentric as

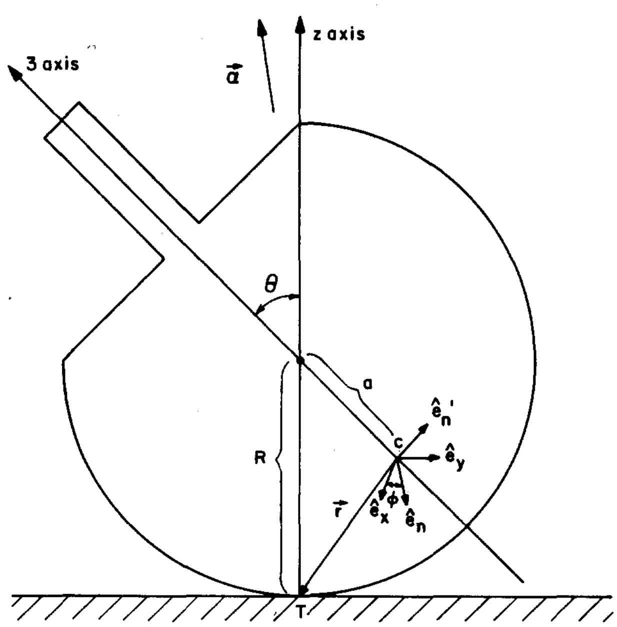

Fig. 3. Tippe top.

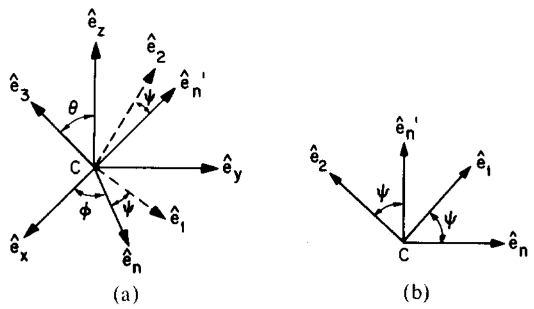

Fig. 4. Coordinate axes.

indicated in Fig. 3. The $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ coordinate system is fixed in the top's body with $\hat{e}_3$ parallel to the stem. The $\hat{e}_x$, $\hat{e}_y$, $\hat{e}_z$ system is fixed in the laboratory frame with $\hat{e}_z$ pointing up, perpendicular to the table surface. We may now proceed to define the $\hat{e}_n$, $\hat{e}_{n'}$, $\hat{e}_3$ coordinate system via the following equations:

$$\hat{e}_n = \hat{e}_z x \hat{e}_3 / |\hat{e}_z x \hat{e}_3|, \quad (2)$$

$$\hat{e}_{n'} = \hat{e}_3 x \hat{e}_n. \quad (3)$$

The unit vectors $\hat{e}_n$, $\hat{e}_{n'}$ remain fixed in the plane in which the vectors $\hat{e}_1$, $\hat{e}_2$ rotate (Fig. 4). Utilizing the Euler angles θ, ϕ, ψ as defined in the figures and Table I, we may write the transformation equations relating the $\hat{e}_x$, $\hat{e}_y$, $\hat{e}_z$, and $\hat{e}_n$, $\hat{e}_{n'}$, $\hat{e}_3$ coordinate systems:

$$\hat{e}_n = \cos\phi \hat{e}_x + \sin\phi \hat{e}_y, \quad (4)$$

$$\hat{e}_{n'} = -\cos\theta \sin\phi \hat{e}_x + \cos\theta \cos\phi \hat{e}_y + \sin\theta \hat{e}_z, \quad (5)$$

$$\hat{e}_3 = \sin\theta \sin\phi \hat{e}_x - \sin\theta \cos\phi \hat{e}_y + \cos\theta \hat{e}_z, \quad (6)$$

$$\hat{e}_x = \cos\phi \hat{e}_n - \cos\theta \sin\phi \hat{e}_{n'} + \sin\theta \sin\phi \hat{e}_3, \quad (7)$$

$$\hat{e}_y = \sin\phi \hat{e}_n + \cos\theta \cos\phi \hat{e}_{n'} - \sin\theta \cos\phi \hat{e}_3, \quad (8)$$

$$\hat{e}_z = \sin\theta \hat{e}_{n'} + \cos\theta \hat{e}_3. \quad (9)$$

We may now express the angular velocity ω of the top as measured in the laboratory frame in terms of the $\hat{e}_n$, $\hat{e}_{n'}$, $\hat{e}_3$ coordinates:

$$\omega = \dot{\theta}\hat{e}_n + \dot{\varphi}\hat{e}_z + \dot{\psi}\hat{e}_3,$$

$$\omega \equiv \omega_n \hat{e}_n + \omega_{n'}\hat{e}_{n'} + \omega_3 \hat{e}_3$$
$$= \dot{\theta}\hat{e}_n + \dot{\phi}\sin\theta \hat{e}_{n'} + (\dot{\psi} + \dot{\phi}\cos\theta)\hat{e}_3. \quad (10)$$

Similarly we may express the angular velocity α of the $\hat{e}_n$,$\hat{e}_{n'}$,$\hat{e}_3$ coordinate system:

$$\alpha = \dot{\theta}\hat{e}_n + \dot{\varphi}\hat{e}_z,$$

$$\alpha \equiv \alpha_n \hat{e}_n + \alpha_{n'}\hat{e}_{n'} + \alpha_3 \hat{e}_3$$
$$= \dot{\theta}\hat{e}_n + \dot{\phi}\sin\theta \hat{e}_{n'} + \dot{\phi}\cos\theta \hat{e}_3. \quad (11)$$

As any observer will readily confirm $\dot{\phi} \gg \dot{\theta}$, for the top rapidly precesses as it slowly turns over. Hence, α points almost totally along the positive z axis. Also, $\dot{\phi}$, the precessional velocity, is substantially larger in magnitude than $|\dot{\psi}|$. The small value of $|\dot{\psi}|$ in the tippe top compared to the conventional top, is partly explainable in terms of the relatively large moment arm—$R \sin\theta$—about which the frictional force acts to retard rotation about the symmetry axis. In the conventional top the frictional force acts with a moment arm equal only to the radius of the axle (Fig. 2). The fact that $\dot{\phi} \gg |\dot{\psi}|$ is equivalent to stating that L_z remains the dominant component of angular momentum throughout.

Table I. Symbols used in text.

R	radius of sphere
C	center of mass of tippe top
a	distance from center of sphere to the center of mass
T	point of contact of the sphere with the table surface
$\mathbf{r}$	position vector from center of mass to point T
$\hat{e}_x,\hat{e}_y,\hat{e}_z$	coordinate system fixed in lab frame with $\hat{e}_z$ perpendicular to table surface
$\hat{e}_1,\hat{e}_2,\hat{e}_3$	coordinate system fixed in top's body with origin at C and with $\hat{e}_3$ parallel to the stem
$\hat{e}_n,\hat{e}_{n'}$	unit vectors defined in text
ψ	angle between $\hat{e}_1$ and $\hat{e}_n$
θ	angle between $\hat{e}_3$ and $\hat{e}_z$
$\dot{\theta}$	nutational velocity
ϕ	angle between $\hat{e}_x$ and $\hat{e}_n$
$\dot{\phi}$	precessional velocity
I_3	principal moment of inertia about $\hat{e}_3$
I	value of either of the other two principal moments of inertia
ω	total angular velocity of $\hat{e}_1,\hat{e}_2,\hat{e}_3$ coordinate system as measured in laboratory frame
α	total angular velocity of $\hat{e}_3,\hat{e}_n,\hat{e}_{n'}$ coordinate system as measured in laboratory frame
μ	coefficient of sliding friction
$\mathbf{F}_f$	frictional force induced at point of contact
$\mathbf{F}_N$	normal force exerted by table surface at point of contact along the $\hat{e}_z$ axis
$\mathbf{L}$	total angular momentum
$\mathbf{N}$	torque about center of mass of top
g	acceleration of gravity
m	mass of top
x_C,y_C,z_C	coordinates of the center of mass as measured in laboratory frame

Equipped with this terminology, we may now present a simple, semiquantitative physical argument that explains why the top turns over on its rounded head. The essentials of this argument were first presented by Pliskin[8]:

The frictional force applied at the point T opposes the motion of the point T in the plane of the table surface. If we ignore for the moment the translational motion of the top, then this force $\mathbf{F}_f$ would point into the paper along the $-\hat{e}_n$ axis. This force leads to a torque $\mathbf{N}_f$ about the center of mass:

$$\mathbf{N}_f = \mathbf{r} \times \mathbf{F}_f$$
$$= (a\hat{e}_3 - R\hat{e}_z) \times (-|\mathbf{F}_f|\hat{e}_n)$$
$$= |\mathbf{F}_f| [(R\cos\theta - a)\hat{e}_{n'} - R\sin\theta \hat{e}_3]. \quad (12)$$

Since the torque due to the normal force acts about the $\hat{e}_n$ axis, this frictional torque $\mathbf{N}_f$ is the only torque with components about the $\hat{e}_{n'}$ and $\hat{e}_3$ axes. Now, if $\theta < \cos^{-1}(a/R)$ the torque about $\hat{e}_{n'}$ is positive and the torque about $\hat{e}_3$ is negative. Hence, we expect $\alpha_{n'}$ to increase as a function of time and α_3 to decrease. This implies that the ratio $\alpha_3/\alpha_{n'} = \cot\theta$ [from Eq. (11)] is monotonically decreasing with time; hence θ will be monotonically increasing. In the range $\cos^{-1}(a/R) \le \theta \le \pi/2$ the torques about $\hat{e}_{n'}$ and $\hat{e}_3$ are both negative, but the torque about $\hat{e}_3$ is the larger in magnitude; hence, providing the two moments of inertia are roughly equal, α_3 should decrease more rapidly than $\alpha_{n'}$, and θ should continue to increase.[9] In the range $\theta \ge \pi/2$ both $\alpha_{n'}$

and α_3 must decrease as a function of time, α_3 becoming more negative since α_3 passes through zero when $\theta = \pi/2$. Since $\alpha_3 = \dot{\phi}\cos\theta$ and $\alpha_{n'} = \dot{\phi}\sin\theta$, the only way both angular velocities can continue to decrease is for θ to continue increasing until the stem of the top touches the table surface. Then a completely analogous analysis rationalizes the top's ability to continue to rise. A detailed analysis of the transfer of the weight of the top to its stem is presented by Hugenholtz.[10]

Presented below is a rigorous analysis of the motion of the top on its rounded head—developed by the author—together with computer-generated solutions of the equations of motion. The force $\mathbf{F}_f$ arising from sliding friction will tend to retard the motion of the point T in the plane of the table. Hence, we must analyze the kinematics of the problem in order to obtain an expression for the velocity of the point T in terms of the other parameters. Let $\mathbf{v}_{TC}$ represent the velocity of the point T with respect to the center of mass of the top:

$$\begin{aligned}\mathbf{v}_{TC} &= \boldsymbol{\omega} \times \mathbf{r} = [(\dot{\psi} + \dot{\phi}\cos\theta)\hat{e}_3 + \dot{\phi}e_n + \dot{\phi}\sin\theta\hat{e}_{n'}] \\ &\quad \times [(a - R\cos\theta)\hat{e}_3 - R\sin\theta\hat{e}_{n'}] \\ &= (R\dot{\psi} + a\dot{\phi})\sin\theta\hat{e}_n + \dot{\theta}(R\cos\theta - a)\hat{e}_{n'} - \dot{\theta}R\sin\theta\hat{e}_3.\end{aligned} \tag{13}$$

Let $\mathbf{v}_{TCP}$ represent the projection of $\mathbf{v}_{TC}$ on the xy plane. Using the transformation Eqs. (4)–(6)

$$\begin{aligned}\mathbf{v}_{TCP} = &[(R\dot{\psi} + a\dot{\phi})\sin\theta\cos\phi + \dot{\theta}\sin\phi(a\cos\theta - R)]\hat{e}_x \\ &+ [(R\dot{\psi} + a\dot{\phi})\sin\theta\sin\phi + \dot{\theta}\cos\phi(R - a\cos\theta)]\hat{e}_y.\end{aligned} \tag{14}$$

Now, if we let $\mathbf{u}_{CP}$ represent the projection of the velocity of the center of mass, $\mathbf{u}_C$, on the xy plane, then we may express $\mathbf{v}_{TP}$, the projection of the velocity of the point T on the xy plane as measured in the laboratory frame:

$$\mathbf{v}_{TP} = \mathbf{u}_{CP} + \mathbf{v}_{TCP}. \tag{15}$$

This prepares us to write the translational equations of motion. Friction acts to oppose the motion of the point T, and the magnitude of the force of sliding friction is proportional to the normal force exerted by the table at the point T[11]:

$$\mathbf{F}_f = (-\mathbf{v}_{TP}/|\mathbf{v}_{TP}|)\mu|\mathbf{F}_N|. \tag{16}$$

Since $\mathbf{F}_f$ is the only external force to act in the xy plane,

$$\mathbf{F}_f = m\dot{\mathbf{u}}_{CP}. \tag{17}$$

The normal force $\mathbf{F}_N$ and the force of gravity act along the z axis and account for the acceleration of the center of mass:

$$|\mathbf{F}_N| - mg = m\ddot{z}_C, \tag{18}$$

where z_C is the z coordinate of the center of mass,

$$z_C = R - a\cos\theta.$$

Hence,

$$\ddot{z}_C = a(\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta). \tag{19}$$

Now we may write the angular equations of motion. Let $\mathbf{F}_T$ represent all forces applied at the point T:

$$\mathbf{F}_T = \mathbf{F}_N + \mathbf{F}_f. \tag{20}$$

The only remaining external force is gravity which acts through the center of mass and hence does not contribute to the total torque $\mathbf{N}$ about the center of mass:

$$\mathbf{N} = \mathbf{r} \times \mathbf{F}_T. \tag{21}$$

It is convenient to write Euler's equations of motion in the $\hat{e}_n$, $\hat{e}_{n'}$, $\hat{e}_3$ coordinate system:

$$\mathbf{N} = \frac{d\mathbf{L}}{dt} = \frac{D\mathbf{L}}{Dt} + \boldsymbol{\alpha} \times \mathbf{L}, \tag{22}$$

where $D\mathbf{L}/Dt$ represents the rate of change of $\mathbf{L}$ as measured in the $\hat{e}_n$, $\hat{e}_{n'}$, $\hat{e}_3$ system.

Since

$$\mathbf{L} = \sum_i \hat{e}_i\omega_i I_i, \tag{23}$$

where I_i are the principal moments of inertia,

$$N_k = I_k\dot{\omega}_k + \alpha_i I_j\omega_j - \alpha_j I_i\omega_i, \tag{24}$$

where i,j,k are a cyclic permutation of $n,n',3$. Since the top is by definition symmetric,

$$I \equiv I_n = I_{n'}. \tag{25}$$

Thus, differentiating our expression for ω [Eq. (10)] appropriately, we arrive at the angular equations of motion:

$$N_3 = I_3(\ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta + \ddot{\psi}), \tag{26}$$

$$N_n = I\ddot{\theta} + I_3\dot{\phi}\dot{\psi}\sin\theta + (I_3 - I)\dot{\phi}^2\sin\theta\cos\theta, \tag{27}$$

$$N_{n'} = I\ddot{\phi}\sin\theta + (2I - I_3)\dot{\theta}\dot{\phi}\cos\theta - I_3\dot{\theta}\dot{\psi}. \tag{28}$$

This completes our six equations of motion (three translational and three angular) and enables us in principle to solve for our six coordinates $\theta,\phi,\psi,x_C,y_C,z_C$.

These horribly nonlinear differential equations were solved using a fourth order Runge–Kutta method of a CDC 6400 computer. For simplicity's sake the theoretical top lacked a stem and consisted of a sphere with an eccentric center of mass. (Thus, in practice, the computer-generated solutions would be valid up to the point when the stem touched the table surface.) The equations were solved for a wide variety of initial conditions, and for a range of values of the parameters μ,I,I_3,R,a.

A sample graph of the solutions obtained for $\theta,\dot{\phi},\dot{\psi}$ as functions of time are displayed in Fig. 5. The initial conditions, $\dot{\psi} = 100$ rad/sec, $\dot{\phi} = 0$, $\theta = 0.1$ rad, correspond to the top set spinning about its symmetry axis at a slight angle to the vertical. The top displays unstable behavior during the first ½ sec of motion as $\dot{\psi}$ decreases in value and $\dot{\phi}$ increases. (The sharp spikes in $\dot{\phi}$ and $\dot{\psi}$ are physically less dramatic than they appear because ω_3,

$$\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta,$$

is a relatively smooth function of time owing to the fact that the spikes in $\dot{\phi}$ and $\dot{\psi}$ cancel each other, and $\omega_{n'}$,

$$\omega_{n'} = \dot{\phi}\sin\theta,$$

remains small in magnitude since θ is small during this first ½ sec.) Once $\dot{\phi}$ gains its dominant position with respect to $\dot{\psi}$, θ nutationally increases in value until the top is inverted. The nutations observed demonstrate that physical arguments which purport to show that θ must be a monotonically increasing function of time (at least for the stated initial

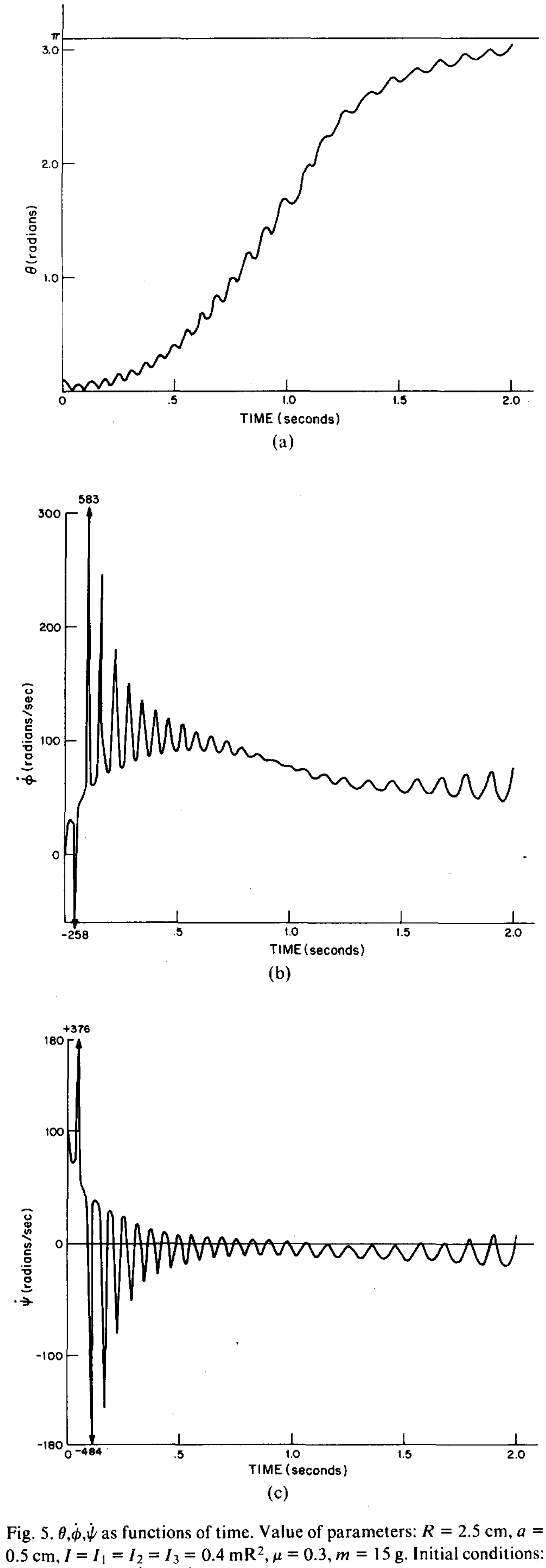

Fig. 5. $\theta, \dot\phi, \dot\psi$ as functions of time. Value of parameters: $R = 2.5$ cm, $a = 0.5$ cm, $I = I_1 = I_2 = I_3 = 0.4\,\mathrm{mR}^2$, $\mu = 0.3$, $m = 15$ g. Initial conditions: $\dot\psi = 100$ rad/sec, $\dot\theta = 0$, $\dot\phi = 0$, $\psi = 0$, $\theta = 0.1$ rad, $\phi = 0$, $\dot x_C = \dot y_C = \dot z_C = 0$.

conditions) cannot be strictly accurate. Indeed, Pliskin also noted that he did experimentally observe nutations, but that these were of marked nature only when the tippe top was spun on a lubricated surface. The fact that the nutations were most noticeable under these circumstances may be explained as follows. We noted that under all "experimental" conditions that $\dot\theta$ oscillates about a mean value that changes slowly in time. When the mean value of $\dot\theta$ is larger than the amplitude of the oscillation, $\dot\theta$ does not go negative. Hence an observer would not notice any true nutations in the sense that θ alternatively increases and decreases in magnitude. Spinning the top on a lubricated surface reduces the size of the friction coefficient μ. Pliskin showed (and we have confirmed with our model) that reducing μ prolongs the inversion time; that is, the mean value of $\dot\theta$ is reduced while the amplitude of the oscillations of $\dot\theta$ is affected to a lesser extent. If the mean value of $\dot\theta$ becomes less than the amplitude of the oscillations, true nutations occur. In Fig. 5 the mean value of $\dot\theta$ is sufficiently small so that true nutations are apparent during most of the inversion process. By increasing μ from 0.3 to 0.8, we succeeded in almost completely abolishing true nutations during most of the inversion process.

Most commercial tippe tops are found to have I_3 smaller than I. We tested Pliskin's hypothesis that if $I_3 < I$ that $\dot\psi$ should initially have the same sign as $\dot\phi$ and that $\dot\psi$ should invert its sign when $\theta = \theta'$, where $\theta' \simeq \pi/2$.[12] We let $I_3 = 0.8I$ and set the simulated top spinning with the same initial conditions as those listed under Fig. 5. We noted that after an initial 0.2-sec period of large fluctuations in the value of $\dot\psi$, that $\dot\psi$ began to oscillate about an average value of +10 rad/sec (Fig. 6). After averaging out the oscillatory component, we found that the mean value of $\dot\psi$ did indeed proceed to go negative when θ was about 1.30 rad. According to Pliskin's calculations the precise value of θ' for this system should be 1.34 rad yielding very good agreement with our results. $\dot\theta$ proceeded to oscillate about a stable value of -11 rad/sec for the remainder of the run. We also confirmed Pliskin's prediction that if $I_3 > I$ that $\dot\psi$ will initially have a sign opposite to that of $\dot\phi$ and that $\dot\psi$ will invert the sign when $\theta = \theta'$. We tested this hypothesis with an "experi-

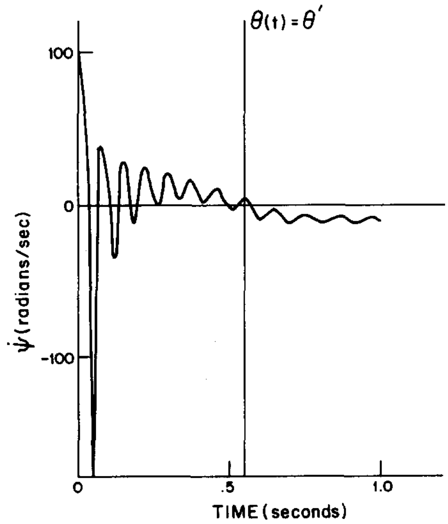

Fig. 6. $\dot\psi$ as function of time. $I_3 = 0.8I$. Other parameters and initial conditions the same as those listed under Fig. 5.

mental" top which had $I_3 = 1.2I$ using the same initial conditions as above except that we had to set $\theta(t = 0) = 0.02$ rad in order to obtain inversion.

We observed some degree of nutation for all values of I_3 that we tested. We found that the smaller the value of I_3, the more rapidly did θ initially increase in value. However, for $I_3 < I$, θ did not fully attain the value of π. For example, for $I_3 = 0.8I$, θ stabilized at a maximum value of about 2.1 rad. Of course, for a real tippe top the stem would touch the table surface well before θ attained the value of π.

In any case, the computer solutions conclusively demonstrate that the action of friction by itself accounts for the inversion of the top, and that no imbalance among any of the moments of inertia need be invoked in order to explain this behavior.

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[3]C. M. Braams, Physica **18**, 503 (1952).

[4]N. M. Hugenholtz, Physica **18**, 515 (1952).

[5]W. A. Pliskin, Am. J. Phys. **22**, 28 (1954).

[6]J. L. Synge, Philos. Mag. **43**, 724 (1952).

[7]A. R. Del Campo, Am. J. Phys. **23**, 544 (1955).

[8]W. A. Pliskin, Ref. 5, p. 30.

[9]At this point in Pliskin's argument, additional clarification is necessary. What one precisely wants to show is that, for $\cos^{-1}(a/R) \leq \theta \leq \pi/2$,

$$0 > \frac{d\cot\theta}{dt} = \left(\frac{\alpha_3}{\alpha_{n'}}\right)\left(\frac{\dot{\alpha}_3}{\alpha_3} - \frac{\dot{\alpha}_{n'}}{\alpha_{n'}}\right)$$

$$= \left(\frac{\cot\theta}{\dot{\phi}}\right)\left(\frac{\dot{\alpha}_3}{\cos\theta} - \frac{\dot{\alpha}_{n'}}{\sin\theta}\right).$$

Since in this range of θ, $\cot\theta/\dot{\phi} > 0$, and since here both $\dot{\alpha}_3$ and $\dot{\alpha}_{n'}$ are presumed less than zero, it is merely necessary to show

$$|\dot{\alpha}_3|/\cos\theta > |\dot{\alpha}_{n'}|/\sin\theta$$

or

$$|\dot{\alpha}_3 \sin\theta/\dot{\alpha}_{n'} \cos\theta| > 1.$$

If we assume

$$N_3/N_{n'} \simeq I_3\dot{\alpha}_3/I_{n'}\dot{\alpha}_{n'} \simeq \dot{\alpha}_3/\dot{\alpha}_{n'},$$

then using Eq. (12)

$$|N_3/N_{n'}| = |R\sin\theta/|(a - R\cos\theta)| \simeq |\dot{\alpha}_3/\dot{\alpha}_{n'}|.$$

Rearranging this expression, we obtain

$$|\dot{\alpha}_3 \sin\theta/(\dot{\alpha}_{n'}\cos\theta)| \simeq |R\sin^2\theta/(a\cos\theta - R\cos^2\theta)|$$

$$= \left|\frac{1-\cos^2\theta}{a\cos\theta/R - \cos^2\theta}\right|$$

$$\geq \left|\frac{1-\cos^2\theta}{(a/R)^2 - \cos^2\theta}\right| > 1.$$

This inequality holds in the range $\cos^{-1}(a/R) \leq \theta \leq \pi/2$.

[10]N. M. Hugenholtz, Ref. 4, p. 525.

[11]This equation is only applicable, of course, if the point of contact is actually sliding (i.e., $\mathbf{v}_{TP} \neq 0$). This was found to be the case for the computer-generated solutions presented below. As $\mathbf{v}_{TP} \to 0$, this expression for $\mathbf{F}_f$ becomes undefined. During rolling, the frictional force may have a magnitude anywhere in the range from 0 to $\mu_{\text{rolling}} \cdot |\mathbf{F}_N|$, and would act so as to oppose the motion the point T would have if the frictional force were instantaneously released. However, numerical computations of the frictional force for all values of $\mathbf{v}_{TP}$ may be performed based on Eq. (16) (assuming μ is the same for rolling and sliding), provided care is used to apply appropriate iterative and limiting procedures.

[12]W. A. Pliskin, Ref. 5, p. 31.