Section 2: Revision of Complex Analysis; Analytic Continuation; Residues

In this lecture we shall review the elements of Complex Analysis. It is important to refresh your memory of this subject by reviewing your third year notes. Here we just remind ourselves of some key-points. We shall also develop the idea of analytic continuation.

2. 1. Whistlestop Tour of Complex Analysis

A function $f(z)$ where $z = x + iy$ is differentiable ($f'(z)$ exists) if $\lim_{\Delta z \to 0} \left[\frac{f(z + \Delta z) - f(z)}{\Delta z} \right]$ is independent of the way in which $\Delta z \to 0$.

This is guaranteed by the 'Cauchy-Riemann conditions' on $f(z) = u(z) + iv(z)$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

A function $f(z)$ is *analytic* in domain D if $f'(z)$ exists through domain D (sometimes this is referred to as $f(z)$ being regular in D).

Cauchy-Goursat theorem (usually just referred to as Cauchy's theorem): if $f(z)$ is analytic within and on a closed contour C

$$\oint_C f(z)\, dz = 0$$

A consequence of this is that $\int_a^b f(z)\, dz$ does not depend on the path.

Cauchy's integral formula: if $f(z)$ is analytic within and on a closed contour C

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$$

Then taking derivatives w.r.t. z_0 one finds

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

Thus analyticity of $f(z)$ within C implies analyticity of all its derivatives - a remarkable result! Functions that are analytic everywhere (except ∞) are *entire* functions e.g. e^x, $\sin x$. This leads us to Liouville's theorem which is if $f(z)$ is analytic $\forall z$ and *bounded* then $f(z)$ is a constant—thus for interesting functions that are bounded we have to be concerned with singularities! (Recall that singularities are places where the function is not analytic)

Series expansions

Taylor's theorem An analytic function has a convergent expansion about any point z within its domain of analyticity.

Proof: $f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z' - z}\, dz'$ where contour C is centred at z_0 and z is within C so $|(z - z_0)/(z' - z_0)| < 1$.

Now

$$\frac{1}{z' - z} = \frac{1}{z' - z_0} \frac{1}{\left(1 - \frac{z - z_0}{z' - z_0}\right)} = \frac{1}{z' - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{z' - z_0} \right)^n$$

and one finds

$$f(z) = \frac{1}{2\pi i}\oint_C \sum_{n=0}^{\infty} \frac{f(z')\ (z-z_0)^n}{(z'-z_0)^{n+1}}\ dz' = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!}\ f^{(n)}(z_0)$$

This gives a unique prescription—the Taylor series— for an expansion about z_0.

Laurent Series

Suppose $f(z)$ has *isolated* singularity at z_0 (but analytic in nbhd of z_0). Then we can write

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \dots$$

We can classify the isolated singularity according to the coefficients b_n:

simple pole	$b_n = 0$ for $n > 1$
pole of order N	$b_n = 0$ for $n > N$
essential singularity	infinitely many powers of $1/(z-z_0)$

Multivalued functions

We will also use the concept of mulitvalued functions e.g. $\ln z$. If we write z in polar form $z = Re^{i\theta}$ then $\ln z = \ln R + i\theta$. Thus for apparently the same point $z = Re^{i\theta + 2\pi i n}$ the function can have different values $\ln z = \ln R + i\theta + 2\pi i n$. In this case an infinite number of values are possible and $\ln z$ has an infinite number of 'branches'. This requires us to introduce 'branch cuts' which are barriers through which z cannot go (e.g. in a line integral).

Aside: Alternatively one can think of Riemann sheets whereby on crossing a branch cut one moves onto a different Riemann sheet of the function; the number of branches equals the number of Riemann sheets. This allows closed contours to be formed by going around branch cuts as many times as required to get back to the original Riemann sheet. However although this scheme is very elegant, for calculational purposes it is best to treat branch cuts as barriers one cannot cross.

2. 2. Analytic Continuation

First let us state an important result

Identity theorem: if two functions are each analytic in a region R and have the same values for all points in some subregion or along some curve within R , the two functions are identical everywhere within R.

We can immediately use this result to extend functions originally defined on the real axis to the complex plane for example $e^z = 1 + z + z^2/2! \dots$ is the *unique* function $f(z)$ that is equal to e^x on the real line.

Analytic Continuation Say we are given a power series about z_1 that has a finite radius of convergence (extending to the nearest singularity). The power series represents a function $f_1(z)$ analytic in the original domain of convergence, at least.

Now consider expanding this function about a new point z_2. The resulting series may converge in a circle extending beyond the original region of convergence.

Using the identity theorem the values $f_2(z)$ in this extended region are uniquely determined by $f_1(z)$ in the common region of convergence. We call $f_2(z)$ the *analytic continuation* of $f_1(z)$ into the new region.

The whole process can then be repeated by expanding $f_2(z)$ about some point in the extended region of convergence to get a function that has a further extended region of convergence (see figure). In principle, in the case of isolates singularities (see caveats below), one can analytically continue along a path to any point on the complex plane. However one doesn't

Figure 1: Illustration of analytic continuation along a path round a singularity

usually carry out this procedure, instead some short cuts and various tricks are used

Example Consider $f_1(z) = \sum_{n=0}^{\infty} z^n$ which converges for $|z| < 1$ to $f(z) = 1/(1-z)$. Now $f(z)$ is analytic everywhere (except for pole at $z = 1$) so $f(z)$ is the analytic continuation of $f_1(z)$ into the entire plane.

Exceptions: Not all functions can be continued indefinitely: the function may have a 'natural barrier' of singularities through which one cannot pass.

It may also happen that the function obtained by continuation is multivalued. For example we may carry out analytic continuation along some path that returns to the first region of convergence. Then if the path has crossed a branch cut the values of the analytically continued function will not agree with the original value.

The 'principle of analytic continuation' may be invoked in other contexts e.g. in integrals (see tutorials), generalising recurrence relations (see next lecture) ...

2. 3. Residue theorem

$$\oint_C f(z)dz = 2\pi i(\text{ sum of residues within } C)$$

where C is an anticlockwise closed contour

Example 1 $\quad I = \displaystyle\int_0^\infty \frac{dx}{1+x^2}$.

Consider a contour in complex plane running along real axis from $-R$ to R then around a semicircle in upper half plane (sketch this).

Along the large semicircle let $z = Re^{i\theta} \quad 0 < \theta < \pi$

$$\int \frac{dz}{1+z^2} = iR \int_0^\pi d\theta \frac{e^{i\theta}}{1+R^2 e^{2i\theta}} \to 0 \quad \text{as} \quad R \to \infty$$

Thus integral around the closed contour reduces to $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2I$. Now $1/(1+z^2) = 1/(z+i)(z-i)$ has simple poles at $z = \pm i$ and $z = i$ is encircled by the contour giving rise to residue $1/2i$. So $2I = 2\pi i \frac{1}{2i} = \pi$ and $I = \pi/2$.

More generally $\int_C e^{iwx} f(x) dx = 0$ where C is the semicircle at infinity in the upper half plane for $w > 0$, provided the large R behaviour of f is

$$|f(Re^{i\theta})| < 1/R \quad \text{for} \quad w = 0$$
$$|f(Re^{i\theta})| \to 0 \quad \text{for} \quad w > 0 \qquad \text{'Jordan's Lemma'}$$

Example 2 Consider a resistance R and inductance L in series with voltage $V(t)$. Suppose $V(t) = \frac{A}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega t} d\omega$ (as we shall see this later on this is actually a voltage impulse at $t = 0$). Now the current due to an alternating voltage e^{iwt} is $e^{iwt}/(R + i\omega L)$ (where $R + i\omega L$ is the complex impedance). Thus

$$I(t) = \frac{A}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{e^{i\omega t}}{R + i\omega L}.$$

To evaluate this integral

$t < 0$	complete contour in lower half plane	$\Rightarrow I(t) = 0$
$t > 0$	complete contour in upper half plane, encircling a pole	$\Rightarrow I(t) = \frac{A}{L} e^{-Rt/L}$

Example 3 $\quad I = \int_0^{\infty} \frac{dx}{1+x^3}$.

Here we cannot close contour in same way as example 1 because the integrand is not even. Instead we use a trick of introducing a branch cut by changing variables to $u = x^3$ then $\frac{dx}{du} = \frac{1}{3} u^{-2/3}$ and

$$I = \int_0^{\infty} \frac{du}{3} \frac{u^{-2/3}}{1+u} .$$

The integrand has a branch point at $u = 0$ and we take a branch cut from $0 \to \infty$ which we must not cross. Just above the branch cut $u = re^{i\epsilon}$ $\epsilon \to 0$ and integrating just above branch cut from $r = 0$ to ∞ gives I. Just below the branch cut $u \to re^{2\pi i}$ and integrating just below the branch cut from $r = \infty$ to 0 gives

$$\int_{\infty}^{0} \frac{dr\ r^{-2/3}}{3} \frac{e^{-4\pi i/3}}{1+r} = -e^{-4\pi i/3} I$$

Then closing the contour by large semicircles (sketch the final contour) we encircle a pole at $u = -1 = e^{i\pi}$. Thus by the residue theorem

$$(1 - e^{-4\pi i/3}) I = 2\pi i \left[e^{-2\pi i/3} \right] /3 .$$

and finally

$$I = \frac{2\pi i}{3.\ 2i\ \sin(2\pi/3)} = \frac{2\pi}{3\sqrt{3}}$$

Note we should also check that the small circuit around the branch point at $u = 0$ doesn't contribute.