Section 3: Gamma function

In this section we consider continuing the factorial function (defined on the integers) into the complex plane. We shall also derive Stirling's approximation.

3. 1. Factorials and the Gamma function

Recall the definition of the factorial defined for +ve integers

$$n! = n(n-1)(n-2)\cdots 1 \quad \text{with} \quad 0! = 1$$

Note that it obeys $n! = n\,(n-1)!$

Now consider an *Integral Representation.*

$$I(n) = \int_0^\infty e^{-t} t^n \, dt \qquad n \text{ integer} \tag{1}$$

Integration by parts gives

$$\begin{aligned} I(n) &= -\left[e^{-t}t^n\right]_0^\infty + n\int_0^\infty e^{-t}t^{n-1}dt \\ \Rightarrow I(n) &= nI(n-1) \\ \text{and} \qquad I(0) &= \int_0^\infty e^{-t}dt = 1 \quad \text{thus} \quad I(n) = n! \end{aligned}$$

Now consider continuing the integral $I(n)$ to the complex plane i.e. replacing n by a complex number z. We require $Re(z) > -1$ so that (1) converges, otherwise the $t = 0$ limit of integration would cause a divergence. Thus we define

$$\Gamma(z+1) \equiv \int_0^\infty e^{-t}t^z dt \qquad Re(z) > -1 \tag{2}$$

$$\text{which (for } Re(z) > 0\text{) again satisfies} \qquad \Gamma(z+1) = z\Gamma(z) \tag{3}$$

This function is known as 'the Gamma function'. Note the unfortunate convention that $\Gamma(n+1) = n!$.

So far (2) only holds for $Re(z) > -1$ and we can't continue it to the entire complex z plane since the integral will diverge. However (3) *can* be continued to $Re(z) < -1$ (except for the points $0, -1, -2 \ldots$ since (3) would imply $1 = \Gamma(1) = 0\Gamma(0)$ and $\Gamma(0) = -\Gamma(-1)$ etc).

To continue (2) we replace it by a contour integral. First note that for z non-integer, t^z will have a branch point at $t = 0$ and we take branch cut $0 \to \infty$ along the positive real axis. We consider 'Hankel's Contour' which is illustrated in the figure (left for you!) We now evaluate the integral along the contour retaining $Re(z) > -1$.

$$\int_{I+II+III} e^{-t}t^z dt \qquad Re(z) > -1$$

We take $t = re^{i\epsilon}$ $\;\epsilon \to 0$ just above the real axis (piece I of contour) i.e. along I the phase of t is zero. Thus the contribution from piece I is

$$\int_\infty^0 e^{-r}r^z dr = -\Gamma(z+1)\ .$$

Figure 1: Hankel's contour which goes around the branch cut

Along piece III we take $t = re^{2\pi i}$. Thus the contribution from piece III is

$$e^{2\pi i} \int_0^\infty e^{-r} r^z e^{2\pi i z} dr = e^{2\pi i z} \Gamma(z+1) \ .$$

Finally along piece II (the small circle encircling the origin) we let $t = \delta e^{i\theta} \quad \theta = 0 \ldots 2\pi$ then $dt = itd\theta$ and the contribution from piece II is

$$i \int_0^{2\pi} e^{-\delta e^{i\theta}} \delta^{z+1} e^{i\theta(z+1)} d\theta \to 0 \quad \text{as} \quad \delta \to 0 \ .$$

Thus we have shown

$$\int_C e^{-t} t^z dt = (e^{2\pi i z} - 1)\Gamma(z+1) \ .$$

or we can rewrite this (multiplying by $e^{-i\pi z}$) as

$$2i \sin \pi z \, \Gamma(z+1) = \int_C e^{-t} (-t)^z dt \tag{4}$$

But along Hankel's contour C, the integral is well-defined for all z since the contour avoids the origin. Thus (4) gives the analytic continuation to $Re(z) < -1$. Sketch the function along the real axis in figure 2.

N.B. For $z = 1, 2 \ldots$ (4) gives 0=0 and one should just use the real integral (1).

3. 2. Beta function

Consider a product of two Gamma functions (with $r > 0$, $s > 0$)

$$\Gamma(r)\Gamma(s) = \int_0^\infty x^{r-1} e^{-x} dx \int_0^\infty y^{s-1} e^{-y} dy \ .$$

Let $x + y = u$

$$\Gamma(r)\Gamma(s) = \int_0^\infty du \int_0^u dx \, x^{r-1} (u-x)^{s-1} e^{-u} \ ,$$

Figure 2: The factorial function extended to negative arguments

then let $x = ut$

$$\begin{aligned}\Gamma(r)\Gamma(s) &= \int_0^\infty du\; e^{-u} u^{r+s-1} \int_0^1 dt\; t^{r-1}(1-t)^{s-1} \\ &= \Gamma(r+s)B(r,s)\end{aligned}$$

$$\text{where the Beta function} \qquad B(r,s) = \frac{\Gamma(r)\,\Gamma(s)}{\Gamma(r+s)} = \int_0^1 dt\; t^{r-1}(1-t)^{s-1} \tag{5}$$

Note that this integral representation is valid for $Re(r) > 0$, $Re(s) > 0$.

Now consider

$$\Gamma(z)\Gamma(1-z) = \Gamma(1)B(z, 1-z) = \int_0^1 dt\; t^{z-1}(1-t)^{-z} = \int_0^\infty dx\; \frac{x^{z-1}}{1+x}$$

where we changed variables $t = x/(1+x)$. The final integral can be evaluated by contour integration (see tutorial) and we obtain the result

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \tag{6}$$

This is sometimes known as Euler's reflection formula. We have derived this result for $0 < Re(z) < 1$ but we know lhs can be continued into entire complex plane as can the rhs therefore by 'the principle of analytic continuation' the result holds $\forall z$ (except integer z).

Setting $z = 1/2$ in (6) yields $\Gamma(1/2) = \sqrt{\pi}$.

3. 3. Stirling's Formula

Stirling's approximation for $\ln N!$ is an important approximation that you have used widely in Statistical Mechanics. Here we derive it as an asymptotic expansion of the Gamma function.

Consider $\Gamma(x+1)$ for $x > 0$ real then we can use the integral representation

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x \; dt$$

Let us change variables to $t = sx$ (we'll see why in a moment) then

$$\Gamma(x+1) = x^{x+1} \int_0^\infty \exp(-sx)\, s^x \; ds = x^{x+1} \int_0^\infty \exp(x\,[-s + \ln s])\; ds$$

The final integrand has the form $\exp x\phi(s)$. For $x \gg 1$ the integrand will be sharply peaked about the maximum of ϕ and we expect the only relevant contributions to the integral to come from the neighbourhood of the maximum.

Now

$$\begin{aligned}
\phi(s) &= -s + \ln s \\
\phi'(s) &= -1 + \frac{1}{s} \quad \text{thus} \quad \phi'(s) = 0 \quad \text{at} \quad s = 1 \\
\phi''(s) &= -\frac{1}{s^2} \qquad \phi'''(s) = \frac{2}{s^3} \qquad \text{and so on}
\end{aligned}$$

We expand about $s = 1$

$$\phi(s) = \phi(1) + (s-1)\phi'(1) + \frac{(s-1)^2}{2}\phi''(1) + \cdots$$

(note that $(s-1)$ term vanishes and $\phi''(1) = -1 < 0$) and change variable to $u = s - 1$ then

$$\Gamma(x+1) = x^{x+1} \int_{-1}^\infty \exp(x\left[-1 - u^2/2 + \cdots\right])\; du$$

We now make two manoeuvres which will justify later. Firstly we drop the higher order terms in u. Secondly we extend the lower limit of the integral to $-\infty$. This results in a gaussian integral that we can perform!

$$\begin{aligned}
\Gamma(x+1) &\simeq x^{x+1} e^{-x} \int_{-\infty}^\infty \exp(-xu^2/2) du \\
&= x^{x+1/2} e^{-x} \sqrt{2\pi}
\end{aligned} \tag{7}$$

Taking the log of (7) we find

$$\ln x! \simeq x \ln x - x + \frac{1}{2}\ln x + \frac{1}{2}\ln(2\pi)$$

This is Stirling's approximation (we usually only need first two terms)

Returning to (7) we note that it is the first term in an asymptotic expansion. We would obtain higher order terms in the expansion by expanding to higher order terms in u. We will examine this more systematically next section but for the moment just state that the next term is smaller by a factor of x:

$$\Gamma(x+1) = x^{x+1/2} e^{-x} \sqrt{2\pi}\left[1 + \frac{1}{12x} + O(1/x^2)\right]$$

and we indeed have an asymptotic expansion.

Our second sleight of hand of extending the limit of integration to infinity is actually quite harmless—it only introduces an error that is exponentially small in x. To see this let us go back and consider

$$\int_{-1}^\infty \exp(-xu^2/2) = \sqrt{\frac{2}{x}} \int_{-\sqrt{x/2}}^\infty \exp(-y^2) dy = x^{x+1/2} e^{-x} \sqrt{\frac{\pi}{2}}\left[1 + \text{erf}(\sqrt{x/2})\right]$$

and for large x, $\text{erf}(\sqrt{x/2}) \sim 1 - e^{-x/2}/\sqrt{\pi x}$ as you are invited to show in tutorial 1.4.