Section 4: Asymptotic expansions of integrals

4. 1. Laplace's Method

In the last section we derived Stirling's approximation by an approach known that is known as 'Laplace's Method'. This is a general method for integrals along the real axis of the form

$$I(x) = \int_a^b f(t) e^{x\phi(t)}\, dt \qquad \text{for} \quad x \gg 0 \tag{1}$$

Idea: there are three steps

(1) if ϕ has a max at $t = c$ then only the immediate nbhd of $t = c$ contributes to the integral (assuming $a < c < b$) i.e.

$$I(x) \simeq \int_{c-\epsilon}^{c+\epsilon} f(t) e^{x\phi(t)}\, dt \qquad \text{for} \quad \epsilon \quad \text{arbitrary +ve}$$

(2) expand $f(t)$ and $\phi(t)$ in series about $\phi(c) \longrightarrow$ series of integrals

(3) extend integration region of each integral to infinity $\longrightarrow$ Gaussian integrals which can be done and from which we obtain an asymptotic expansion

Often we are just interested in the first term of the expansion. Using the above method we: (1) identifty c ; (2) set

$$f(t) \simeq f(c) \qquad \phi(t) \simeq \phi(c) + \frac{(t-c)^2}{2}\phi''(c)\ldots$$

(why no linear term in $t - c$ for ϕ?); then (3)

$$\begin{aligned} I(x) &\simeq \int_{-\infty}^{\infty} f(c) \exp\{x\phi(c) - x|\phi''(c)|(t-c)^2/2\} dt \\ &= f(c) \exp(x\phi(c)) \int_{-\infty}^{\infty} du\, e^{-x|\phi''(c)|u^2/2} \\ &= f(c) \exp(x\phi(c)) \sqrt{\frac{2\pi}{|\phi''(c)|}}\, \frac{1}{x^{1/2}} \end{aligned} \tag{2}$$

If we wish to calculate the full series we should expand $f(t)$ and $\phi(t)$ to higher order i.e,

$$\begin{aligned} I(x) \simeq \int_{-\epsilon}^{\epsilon} &\left[f(c) + uf'(c) + \frac{u^2}{2} f''(c) + \cdots \right] \\ &\times \exp\left\{ x\phi(c) - x|\phi''(c)|u^2/2 + x\phi^{(3)}(c)u^3/3! + x\phi^{(4)}(c)u^4/4! + \cdots \right\} du \end{aligned}$$

where $\phi^{(n)}$ means nth derivative of ϕ. The key step is now to expand the higher order terms in the exponential as follows

$$\begin{aligned} &\exp\left\{ x\phi^{(3)}(c)u^3/3! + x\phi^{(4)}(c)u^4/4! + \cdots \right\} = \\ &\quad \left[1 + \left(x\phi^{(3)}(c)u^3/3! + x\phi^{(4)}(c)u^4/4! + \cdots \right) + \frac{1}{2} \left(x\phi^{(3)}(c)u^3/3! + x\phi^{(4)}(c)u^4/4! + \cdots \right)^2 + \cdots \right] \end{aligned}$$

then we can extend the limit of the integrals to infinity and end up with a plethora of Gaussian integrals to compute.

The general formula is

$$\int_{-\infty}^{\infty} du\, u^n\, e^{-au^2/2} = \begin{cases} 0 & \text{if} \quad n \quad \text{odd} \\ \frac{\sqrt{2\pi}}{a^{(n+1)/2}}(n-1)(n-3)(n-5)....(3)(1) & \text{if} \quad n \quad \text{even} \end{cases} \tag{3}$$

And it remains to collect terms of the same order in x. For example the first correction to (2) is

$$\exp(x\phi)\sqrt{\frac{2\pi}{|\phi''|}}\,\frac{1}{x^{3/2}}\left[-\frac{f^{(2)}}{2\phi^{(2)}}+\frac{f\phi^{(4)}}{8(\phi^{(2)})^2}+\frac{f^{(1)}\phi^{(3)}}{2(\phi^{(2)})^2}-\frac{5f(\phi^{(3)})^2}{24(\phi^{(2)})^3}\right] \tag{4}$$

where f, ϕ and derivatives are all evaluated at c. As you can guess this procedure (has) soon become(s) very tedious! But the important point for us is that it shows we develop an asymptotic expansion:

$$I(x) \sim \exp(x\phi(c))\sqrt{\frac{2\pi}{|\phi''(c)|}}\,\frac{1}{x^{1/2}}\left[f(c)+\frac{A}{x}+\frac{B}{x^2}+\cdots\right].$$

Further Notes

- To bring an integral into the form (1) we sometimes have to knock it into shape by changing variable etc as we did when we derived Stirling's approximation
- it may happen that there is no maximum of ϕ in the interval (a,b) then the integral will be dominated by one of the endpoints. In this case we have to be more careful with our expansion about a or b since ϕ' generally doesn't vanish (see tutorial 2.8)
- if there are several maxima in the interval we should consider the highest

4. 2. Method of Stationary Phase

Consider

$$I(x) = \int f(t)e^{ix\psi(t)}\,dt \qquad \text{for} \quad x \gg 0 \tag{5}$$

When x is large the integrand will oscillate very rapidly due to the phase factor $e^{ix\psi(t)}$. Thus if $f(t)$ is smooth we expect the contributions to the integral to nearly cancel. But near a stationary point the oscillation is less rapid since the phase is stationary. Thus we expect the dominant contribution to the integral to come from near $\psi'(t) = 0$.

Then one proceeds analogously to Laplace's method and the leading order approximation is

$$I(x) = f(c)\exp(ix\psi(c))\int_{-\infty}^{\infty} du\, e^{ix\psi''(c)u^2/2}$$

We then use the result (see tutorial)

$$\int_{-\infty}^{\infty} e^{i\alpha u^2}\,du = \sqrt{\frac{\pi}{\alpha}}e^{i\pi/4}$$

to obtain

$$
\begin{aligned}
I(x) &\sim f(c) e^{ix\psi(c)} \sqrt{\frac{2\pi}{x\,\psi''(c)}}\, e^{i\pi/4} \\
&= f(c) e^{ix\psi(c) \pm i\pi/4} \sqrt{\frac{2\pi}{|\psi''(c)|}}\, \frac{1}{x^{1/2}}
\end{aligned}
$$

where the $\pm$ is according to whether c is a min or a max of $\psi(t)$

Example: Airy integral for large, negative argument

$$
Ai(-x) = \frac{1}{\pi} \int_0^\infty d\omega \cos\left(\frac{\omega^3}{3} - x\omega\right) \qquad x > 0
$$

First note that we may write the integral as

$$
\begin{aligned}
Ai(-x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \exp\left[i\left(\frac{\omega^3}{3} - x\omega\right)\right] \\
&= \frac{x^{1/2}}{2\pi} \int_{-\infty}^{\infty} dz \exp\left[ix^{3/2}\left(\frac{z^3}{3} - z\right)\right] \qquad (\omega = x^{1/2} z)
\end{aligned}
$$

The stationary points of $\psi(z) = \dfrac{z^3}{3} - z$ are $z = \pm 1$ at which $f''(z) = 2z = \pm 2$. In this case we have two stationary points, the contributions from which we must sum. Expanding about the stationary points to second order and performing the integral yields

$$
\begin{aligned}
Ai(-x) &\simeq \frac{x^{1/2}}{2\pi} \sum_{\pm} \int_{-\infty}^{\infty} dz \exp ix^{3/2} \left(\mp 2/3 \pm (z \mp 1)^2\right) \\
&= \frac{x^{1/2}}{2\pi} \sum_{\pm} \exp(\mp 2ix^{3/2}/3) \left(\frac{\pi}{x^{3/2}}\right)^{1/2} e^{\pm i\pi/4} \\
&= \frac{1}{\sqrt{\pi} x^{1/4}} \cos(2x^{3/2}/3 - \pi/4)
\end{aligned}
$$

The method of stationary phase arises in the 'WKB' approach to Schrödinger's equation (see later) and in path integral formulation of quantum mechanics.