Section 5: The saddle-point method a.k.a. 'method of steepest descents'

Laplace's method and the method of stationary phase are just two instances of a general procedure known as the saddle-point method. Here we are concerned with an integral in the complex plane of the form

$$I(N) = \int_a^b g(z) e^{Nf(z)}\, dz \qquad \text{for} \quad N \gg 0 \tag{1}$$

where f is now an analytic complex function.

We expect the integral to be dominated by the highest stationary points of f. To argue this we combine the arguments for Laplace's method and stationary phase i.e. if $f = u + iv$ we expect integral to be dominated by points where u is maximised and also we require that v is stationary so that the oscillating contributions do not cancel. Thus we require $f'(z) = 0$.

Actually the only extrema possible for f or $Re(f)$ are *saddle points.* To see this recall that for $f = u+iv$ the Cauchy-Riemann conditions imply that both u, v satisfy Laplace's equation i.e.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Thus at a stationary point if, for example, $\dfrac{\partial^2 u}{\partial x^2} > 0$ then $\dfrac{\partial^2 u}{\partial y^2} < 0$

We expect the integral to be dominated by the highest saddle point. Say this is at z_0. To evaluate (1) we deform the contour (as we may by Cauchy's theorem) so that it passes through the saddle point. Near z_0 we set

$$f(z) \simeq f(z_0) + \frac{1}{2} f''(z_0)(z - z_0)^2 \qquad g(z) \simeq g(z_0)$$

and $I(N)$ becomes

$$I(N) = g(z_0) e^{Nf(z_0)} \int \exp\left\{ \frac{1}{2} N f''(z_0)(z - z_0)^2 \right\} dz\ .$$

Now set

$$z - z_0 = r e^{i\phi} \qquad f''(z_0) = |f''(z_0)| e^{i\theta}$$

We are free to choose ϕ which is the the angle at which the contour passes through the saddle point; we will make a convenient choice of ϕ to make the final integration simple.

We have

$$I \simeq g(z_0) \exp(Nf(z_0)) \int \exp\left\{ \frac{1}{2} N |f''(z_0)| e^{i\theta} r^2 e^{2i\phi} \right\}\ e^{i\phi}\, dr$$

Now choose $\theta + 2\phi = \pi$ i.e. $\phi = (\pi - \theta)/2$ then

$$I \simeq g(z_0) e^{Nf(z_0)}\, e^{i\phi} \int dr \exp\left\{ -\frac{1}{2} N |f''(z_0)| r^2 \right\}$$

Extending as usual the limits of integration to infinity and performing the gaussian integral yields

$$I \simeq g(z_0) e^{Nf(z_0)}\, e^{i\phi} \left(\frac{2\pi}{N |f''(z_0)|} \right)^{1/2} \qquad \text{where} \quad \phi = \frac{\pi - \theta}{2} \tag{2}$$

(2) gives 'the saddle-point approximation' to the integral.

As with Laplace's method one can in principle calculate further terms in the asymptotic series but again this is very tedious.

The path we chose to make the integration a gaussian (i.e. fixed phase $\phi = (\pi - \theta)/2$) corresponds to the path that descends most steeply from the saddle point (see e.g. *Arfken* for proof). Hence the name. Actually taking this path is not essential e.g. the method of stationary phase takes another path (see below) but of course gives the same final result.

5. 1. Relation to Laplace's method and method of stationary phase

Laplace's method corresponds to having a saddle point on the real axis with $\dfrac{\partial^2 u}{\partial x^2} < 0$ (maximum in the x (real) direction) and $\dfrac{\partial^2 u}{\partial y^2} > 0$ (minimum in the y (imaginary) direction). We go through the saddle point along the real axis and $\phi = 0$ i.e. in (2) f'' is real and negative so $\theta = \pi$ and $\phi = 0$.

The method of stationary phase corresponds to going through a saddle point on a contour where $\theta + 2\phi = \pi/2$ so that

$$\int dz \exp\left\{\frac{N}{2} f''(z_0)(z - z_0)^2\right\} = e^{-i\theta/2 + i\pi/4} \int dr \exp\left\{\frac{iN}{2} |f''(z_0)| r^2\right\}$$

5. 2. Example: The binomial coefficient

Consider the following contour integral representation of the binomial coefficient

$$\left(\begin{array}{c} N \\ M \end{array}\right) = \oint \frac{dz}{2\pi i} \frac{(1+z)^N}{z^{M+1}}$$

where the contour encircles the origin. To understand this formula expand $(1+z)^N$ using the binomial expansion then use the residue theroem to show that only the z^M term of the expansion survives through the contour integration. Thus the contour integral 'sifts' the binomial expansion to select the correct term- make sure you understand this. Now consider large M and N. Let $M = Ny$ and write the integrand as

$$\left(\begin{array}{c} N \\ M \end{array}\right) = \oint \frac{dz}{2\pi i} \frac{1}{z} \exp N \left[\ln(1+z) - y \ln z\right]$$

Now identify the saddle point: let $f(z) = \ln(1+z) - y \ln z$

$$\begin{aligned} f'(z) &= \frac{1}{1+z} - \frac{y}{z} \qquad f''(z) = -\frac{1}{(1+z)^2} + \frac{y}{z^2} \\ f'(z_0) &= 0 \quad \Rightarrow z_0 = \frac{y}{1-y} \\ f(z_0) &= -y \ln y - (1-y)\ln(1-y) \qquad f''(z_0) = \frac{(1-y)^3}{y} \end{aligned}$$

Since the only singularity in the original integrand is at $z = 0$ we can deform the contour to pass through the saddle point and use the saddle point formula (2) that we derived in the first part of the section.

$$
\begin{aligned}
\left(\begin{array}{c} N \\ M \end{array} \right) &\simeq \frac{1}{2\pi i} \frac{e^{i\phi}}{z_0} e^{Nf(z_0)} \left(\frac{2\pi}{Nf''(z_0)} \right)^{1/2} \qquad \text{where } \phi = \pi/2 \\
&= \left(\frac{1}{2\pi N y(1-y)} \right)^{1/2} \exp -N \left[y \ln y + (1-y) \ln(1-y) \right]
\end{aligned}
$$

The argument of the exponential should be familiar from statistical mechanics. In this example the saddle point z_0 lies on the real axis and is a minimum along the real axis but a maximum with respect to the imaginary direction. We go through the saddle point in the imaginary direction.

5. 3. Further Notes on the saddle point method

- If there are several saddle points one should sum the contribution from each. Often one will be dominant. Interesting phenomena occurs when on varying a parameter one saddle point rises above the other. Then the asymptotic expression for the integral changes in a non-analytic way. In statistical physics this is the mechanism for a first order phase transition.
- Care should be taken to go through the saddle point in the correct *sense* otherwise one can obtain minus the true approximation to the integral (see next example).
- If there is no saddle point then the integral will be dominated by a boundary where f' may not be zero and a slightly different expansion has to be made.

5. 4. Example: Hankel function

The Hankel functions (of the first kind) are solutions of Bessel's equation (see later). They may be written as a contour integral

$$
H_\nu^{(1)}(x) = \frac{1}{\pi i} \int_{0+i\epsilon}^{-\infty+i\epsilon} \exp\left[\frac{x}{2} \left(z - \frac{1}{z} \right) \right] \frac{dz}{z^{\nu+1}}
$$

where there is a branch cut along the negative real axis. Let $f(z) = \frac{1}{2}\left(z - \frac{1}{z} \right)$. We find there are two saddle points:

$$
f'(z) = \frac{1}{2} + \frac{1}{2z^2} \qquad f'(z_0) = 0 \quad \Rightarrow z_0 = \pm i
$$

Now due to the nature of the contour (sketch the figure) we can deform the contour to go through the saddle point $z_0 = i$. (Think about why we don't deform to go through the other saddle point.)

$$
f''(z) = -\frac{1}{z^3} \quad f''(z_0) = -i = e^{-i\pi/2}
$$

Figure 1: The contour for the Hankel function $H_\nu^{(1)}$, position of saddle points and deformed contour.

Let $z - i = re^{i\phi}$ Then

$$H_\nu^{(1)} \sim \frac{e^{i\phi}}{\pi i} \int \frac{dr}{i^{\nu+1}} \; \exp\left[x\left(i + \frac{r^2}{2} e^{i(2\phi - \pi/2)}\right)\right]$$

To turn the integral into a Gaussian we choose $2\phi - \pi/2 = \pi$ i.e. $\phi = 3\pi/4$. Then

$$\begin{aligned} H_\nu^{(1)}(x) &\sim \frac{e^{3\pi i/4 + ix}}{\pi \, i^{\nu+2}} \int_{-\infty}^{\infty} dr \; \exp(-xr^2/2) \\ &= \exp\left(3\pi i/4 + ix - \pi i(\nu+2)/2\right) \sqrt{\frac{2}{\pi x}} \\ &= \sqrt{\frac{2}{\pi x}} \; e^{i(x - \nu\pi/2 - \pi/4)} \end{aligned}$$

You should check that this agrees with the general result (2).

We now return to the question of the *sense* of the contour. We could also have turned the final integral above into a Gaussian by choosing $\phi = -\pi/4$ which would have yielded -ve the correct final result. The reason is (compare with your sketch) that $\phi = -\pi/4$ would mean going backwards through the saddle point (remember ϕ is the angle of the contour to the real axis). When using the saddle point formula (2) it is best to check that $\phi = (\pi - \theta)/2$ gives the correct sense for passing through the saddle point, according to the original integration contour.