6. 1. Physical examples

Consider an 'impulse' which is a sudden increase in momentum $0 \rightarrow mv$ of an object applied at time t_0 say. To model this in terms of an applied force i.e. a 'kick' $F(t)$ we write

$$mv = \int_{t_0-\tau}^{t_0+\tau} F(t)dt$$

which is dimensionally correct, where $F(t)$ is strongly peaked about t_0 (see figure).

Actually the details of the shape of the peak are not important, what is important is the area under the curve. To o btain the limit of an impulse we would like to make the duration (2τ) of the force smaller but the strength greater such that mv remains the same. Then in the limit $\tau \rightarrow 0$ we obtain an impulse. In this limit we write

$$F(t) = mv\delta(t - t_0)$$

Figure 1: Sketch of force against time to deliver an impulse

Another physical example is a point mass which is a finite mass M concentrated at a point $\underline{r}_0$. Then the mass density $\rho(\underline{r})$ is zero everywhere except $\underline{r}_0$ so we need a strongly peaked function

$$\rho(\underline{r}) = M\delta(\underline{r} - \underline{r}_0) \ .$$

Also we should have $\int_V \rho(\underline{r})d^3r = M$ (total mass $= M$) and to calculate for example a moment of inertia we want $\int_V \rho(\underline{r})r^2 d^3r = Mr_0^2$. Thus generally we require $\int_V \rho(\underline{r})f(\underline{r})d^3r = Mf(\underline{r}_0)$.

Symbolically we can think of the delta function $\delta(x)$ as a spike at $x = 0$

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

but this is not a satisfactory definition. What we really require is

Definition of Dirac delta

$$\int_{-\infty}^{\infty} \delta(x)f(x)dx = \int_{-\epsilon}^{+\epsilon} \delta(x)f(x)dx \quad = \quad f(0) \qquad (1)$$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \qquad (2)$$

6. 2. Delta sequences

Does a function as defined above exist? Unfortunately, not in the usual sense of a function, since a function that is zero everywhere except at a point is not well defined. But there exists sequences of functions that approach the sifting property (1) in a certain limit.

Example: Top hat function

$$\delta_n^{(1)}(x) \equiv \begin{cases} 0 & x < -\frac{1}{n} \\ \frac{n}{2} & -\frac{1}{n} < x < \frac{1}{n} \\ 0 & \frac{1}{n} < x \end{cases}$$

Aside: These functions have *compact support* i.e. non zero only on a finite interval.

If f is continuous at zero it means that $\forall \epsilon > 0 \, \exists \eta$ *s.t.* for $|x| < \eta \quad |f(0) - f(x)| < \epsilon$. It follows that $\forall n > 1/\eta$

$$\left| f(0) - \int_{-\infty}^{\infty} dx \delta_n^{(1)}(x) f(x) \right| = \left| \frac{n}{2} \int_{-1/n}^{1/n} dx \, (f(0) - f(x)) \right| \leq \frac{n}{2} \int_{-1/n}^{1/n} dx \, |f(0) - f(x)| < \epsilon$$

Thus

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} dx \, \delta_n^{(1)}(x) f(x) = f(0)$$

for any continuous function f.

Another sequence which is often very useful for analytic calculation is

$$\delta_n^{(2)}(x) \equiv \frac{n}{\sqrt{\pi}} e^{-n^2 x^2}$$

These functions have the advantage of all derivatives existing everywhere.

Some other useful sequences are

$$\delta_n^{(3)}(x) \equiv \frac{n}{\pi} \frac{1}{1 + n^2 x^2} \qquad \delta_n^{(4)}(x) \equiv \frac{1}{n\pi} \frac{\sin^2 nx}{x^2}$$

With all the above sequences, although the required sifting property is approached in the limit, the limit of the sequence of functions doesn't actually exist—they just get narrower and higher without limit! Thus the 'delta function' only has meaning beneath the integral sign.

6. 3. Integral representation

Recall the 'Heaviside' or 'theta' or 'step' function defined as

$$\theta(t) \equiv \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} \tag{3}$$

This function is clearly discontinuous at $t = 0$ and it is usual to take $\theta(0) = 1/2$.

Now consider the following integration by parts

$$\begin{aligned}\int_{-\infty}^{\infty} dt\, f(t)\theta'(t) &= [f(t)\theta(t)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dt\, f'(t)\theta(t) \\ &= f(\infty) - \int_{0}^{\infty} dt\, f'(t) = f(\infty) - [f(t)]_0^\infty = f(0)\end{aligned}$$

Thus we can identify

$$\delta(t) = \theta'(t)$$

Now we note an integral representation of the theta function

$$\theta(x) = \frac{1}{2\pi i}\int_{-\infty-i\gamma}^{\infty-i\gamma} dk\,\frac{e^{ikx}}{k} \qquad \gamma > 0\ .$$

To understand this identity note that for $x > 0$ we can close contour in upper half plane. We then encircle the pole at the origin and pick up a residue. However for $x < 0$ we close the contour in the lower half-plane and encircle no poles so the integral is zero.

From this we can develop an integral representation of the delta function

$$\delta(x) = \frac{d\theta(x)}{dx} = \int_{-\infty-i\gamma}^{\infty-i\gamma} \frac{dk}{2\pi}\, e^{ikx} \qquad \gamma > 0\ .$$

But now there is no pole at $k = 0$ so we can conveniently take $\gamma = 0$ and write

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} \tag{4}$$

Of course, this integral is actually divergent and only has meaning under another integral sign.

To get some insight into (4) let us recall an integral representation of the Kronecker delta (defined on integers) that we have implicitly used in previous lectures

$$\delta_{mn} = \oint \frac{dz}{2\pi i}\frac{z^m}{z^{n+1}}$$

where the integral encircles the origin. This is a direct consequence of the residue theorem. Now change variables to $z = e^{ik}$ where $-\pi < k < \pi$

$$\delta_{mn} = \int_{-\pi}^{\pi} \frac{dk}{2\pi}\, e^{ik(m-n)} \tag{5}$$

Identities (4) and (5) are of fundamental importance in the theory of Fourier transforms and Fourier series respectively

6. 4. Other properties

Firstly it is easy to show by changing variable that

$$\int_{-\infty}^{\infty} dx\, f(x)\delta(x-y) = f(y) \tag{6}$$

To give meaning to the derivative of the delta function we can integrate by parts

$$\int_{-\infty}^{\infty} dx\, f(x)\delta'(x) = [f(x)\delta(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\, f'(x)\delta(x) = -f'(0) \tag{7}$$

Similarly one can show

$$\int_{-\infty}^{\infty} dx\, f(x)\frac{d^n\delta(x)}{dx^n} = (-1)^n \frac{d^n f(0)}{dx^n} \tag{8}$$

For a change of variable $\delta(g(x))$ i.e. delta function of a function we find

$$\begin{aligned}
\int_{-\infty}^{\infty} dx\, f(x)\delta(g(x)) &= \sum_{x_0} \int_{x_0-\epsilon}^{x_0+\epsilon} dx\, f(x)\delta(g(x)) \qquad \text{where} \quad g(x_0) = 0 \\
&= \sum_{x_0} \int_{g(x_0-\epsilon)}^{g(x_0+\epsilon)} \frac{dy}{g'(x)}\, f(g^{-1}(y))\delta(y) \qquad \text{where} \quad y = g(x) \\
&= \sum_{x_0} \frac{f(x_0)}{|g'(x_0)|}
\end{aligned}$$

Thus

$$\delta(g(x)) = \sum_{x_0} \frac{\delta(x-x_0)}{|g'(x_0)|} \qquad \text{where} \quad x_0 \text{ are zeros of } \quad g(x) \tag{9}$$

We can also extend easily to higher dimensions e.g.

$$\delta(\underline{r} - \underline{r}_0) = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0) \tag{10}$$

*6. 5. *Theory of distributions or generalised functions*

As we have noted the δ function does not exist as a function. Its defining properties (1,2) are both in the form of an integral with some function g

$$\int dx\, \delta(x)g(x) = g(0)$$

The function $g(x)$ is known as a 'test function'. In order to make the delta function respectable we need to define a class of test functions for which the defining properties can be realised. Then going back to our delta sequences we want the sequence of integrals to converge for $g(x)$ within the class of test functions. In this way we can define the delta function as a *distribution* which is an object that acts in a certain way on the class of test functions. The key difference between a distribution and a function is that a distribution is not defined in terms of values at points.

In practice we don't need to worry about all this and we happily refer to the 'Dirac delta function'. But we should take care to avoid pitfalls such as multiplying two distributions (with the same variable) together e.g. $\delta^2(x)$ is not defined.