Section 7: Ordinary Differential Equations

7. 1. Ways of writing of ODEs

An n^{th} *order* ordinary differential equation (o.d.e.) is a relation

$$y^{(n)}(x) = F(x, y(x), y'(x) \dots y^{(n-1)}(x)) \ . \tag{1}$$

The equation is *linear* if F is a linear function of y and its derivatives (but the x dependence can be non-linear).

In general the solution of (1) depends on n independent parameters sometimes called the n constants of integration. If the equation is non-linear then there may be special additional solutions.

A general n^{th} order o.d.e. may also be written as a system of n first order equations i.e Let $y_k = \dfrac{d^k y}{dx^k} \quad k = 0 \dots n-1$ then

$$\begin{aligned} \frac{d\ y_k}{dx} &= y_{k+1}(x) \qquad k = 0 \dots n-2 \\ \frac{d\ y_{n-1}}{dx} &= F(x, \{y_k(x)\}) \end{aligned}$$

Sometimes first order ODEs are written

$$A(x,y)dy + B(x,y)dx = 0 \ .$$

A linear o.d.e. may formally be written as

$$\begin{aligned} \mathcal{L}y(x) &= f(x) & (2) \\ \text{where} \quad \mathcal{L} &= p_0(x) + p_1(x)\frac{d}{dx} + \cdots + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \frac{d^n}{dx^n} & (3) \end{aligned}$$

$\mathcal{L}$ is a linear differential operator, simply meaning that we can add two such operators to get another. The coefficent of the highest order derivative is conventionally taken to be one.

If $f(x) = 0$ the o.d.e. is *homogeneous.*

7. 2. Theory of linear homogeneous ordinary differential equations

The general solution of

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y^{(1)} + p_0(x)y = 0 \tag{4}$$

is $y(x) = \sum_{j=1}^{n} C_j y_j(x)$ where C_j are constants of integration and $y_j(x)$ are the set of n linear independent functions satisfying the o.d.e.

Recall the definition of linear dependent *functions*:

if we can find α_j *s.t.* $\sum_{j=1}^{n} \alpha_j y_j(x) = 0 \quad \forall x$ the n functions are linearly *dependent.* Note that this must hold with the same α_j for all x in the interval. If the only solution is $\alpha_j = 0 \; \forall \; j$ the functions are *independent*.

To test for lin.dep of functions take the definition of linear dependence and differentiate $n-1$ times. This implies n conditions that must be satisfied:

$$\begin{aligned}
\sum_{j=1}^{n} \alpha_j y_j(x) &= 0 \\
\sum_{j=1}^{n} \alpha_j y_j'(x) &= 0 \\
&\vdots \\
\sum_{j=1}^{n} \alpha_j y_j^{(n-1)}(x) &= 0
\end{aligned}$$

Thus we have n linear equations in n unknowns α_j and to have a nontrivial solution we require that the *Wronskian* defined as

$$\begin{aligned}
W(x) &= W\left[y_1(x), y_2(x) \ldots y_n(x)\right] \\
&\equiv \det \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}
\end{aligned} \tag{5}$$

vanish at all points x in the interval. Thus the vanishing of $W(x)$ implies linear dependence of y_j **N.B.** $W(x)$ should vanish identically, not just at specific points.

Now the Wronskian of $W(x)$ of any n solutions of (4) satisfies

$$W'(x) = -p_{n-1}(x)W(x)$$

(the derivation of this is left to a tutorial problem). The solution (defined up to a multiplicative constant) is

$$W(x) = \exp\left[-\int^x p_{n-1}(t)dt\right] \qquad \text{Abel's formula} \tag{6}$$

(Of course $W = 0$ is a solution but that just corresponds to lin. dep. y_j .)

7. 3. Second order linear homogeneous differential equations

The Wronskian is particularly useful for second order ODEs where we have two independent solutions y_1,y_2. Consider

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{y_1 y_2' - y_1' y_2}{y_1^2} = \frac{W(x)}{y_1^2(x)}$$

$$\text{thus} \qquad y_2(x) = y_1(x) \int_{x_0}^{x} \frac{W(t)}{y_1^2(t)}\, dt \tag{7}$$

where x_0 is a constant. Thus if we know one solution of the o.d.e. we can use the Wronskian to compute another linearly independent solution.

Example: Laguerre's equation $xy'' + (1-x)y' + ny = 0$

For $n = 0$ one simple solution is $y_1 = 1$.

$$W = \exp\left[-\int^x \left(\frac{1-t}{t}\right) dt\right] = A \exp\left[-\ln x + x\right]$$

Thus $y_2 = A \int^x \frac{e^{x'}}{x'} dx'$. Note that $y_2(x)$ has a logarithmic singularity at $x = 0$.

7. 4. Second order linear inhomogeneous differential equations— Green functions

Let us write our second order ODE in 'Sturm-Liouville' form

$$\mathcal{L}(x) y(x) = f(x) \tag{8}$$

where

$$\mathcal{L}(x) = \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x)$$

Actually all second order equations can be written in this way i.e.

$$y''(x) + p_1(x)y'(x) + p_0(x)y(x) = g(x)\ .$$

can be rewritten using $\dfrac{dW(x)}{dx} = -p_1(x)W(x)$ as

$$W(x)\left[\frac{d}{dx}\left(\frac{1}{W(x)}\frac{d}{dx}\right) + \frac{p_0(x)}{W(x)}\right] y(x) = g(x)$$

which is in the form (8) when we identify

$$p(x) = \frac{1}{W(x)} \quad q(x) = \frac{p_0(x)}{W(x)} \quad f(x) = \frac{g(x)}{W(x)}\ .$$

Generally (8) is defined in a range $a \le x \le b$ and boundary conditions are typically of form

$$\alpha y(a) + \beta y'(a) = 0 \qquad \gamma y(b) + \delta y'(b) = 0$$

which are *homogeneous* (i.e. the right hand sides of the equations are zero) and *unmixed* (each equation involves just one boundary).

Let us assume that the particular solution corresponding to these boundary conditions may be written in the form

$$\boxed{y_p(x) = \int_a^b dx' G(x, x') f(x') \quad (9)}$$

where $G(x, x')$ is known as the **Green Function**. Inserting this in (8) noting that $\mathcal{L}(x)$ acts on the variable x whereas x' is merely a dummy variable yields

$$\int_a^b dx' \; [\mathcal{L}(x) G(x, x')] \, f(x') = f(x)$$

From this equation we recognise that the term in the square bracket must be a delta function

$$\boxed{\mathcal{L}(x) G(x, x') = \delta(x - x') \quad (10)}$$

To determine $G(x, x')$ we first integrate wrt x over the delta function (with x' as a fixed parameter).

$$\int_{x'-\epsilon}^{x'+\epsilon} \left\{ \frac{d}{dx} \left[p(x) \frac{dG(x, x')}{dx} \right] + q(x) G(x, x') \right\} dx = 1$$

$$\Rightarrow \left[p(x) \frac{dG(x, x')}{dx} \right]_{x'-\epsilon}^{x'+\epsilon} + 2\epsilon \left[q(x) G(x, x') \right] \Bigg|_{|x-x'|<\epsilon} = 1$$

We now **choose** $G(x, x')$ to be continuous at $x = x'$ (and assume $p(x), q(x)$ also continuous) then as $\epsilon \to 0$ the second term on the lhs vanishes and the first term yields

$$p(x') \left[\frac{dG(x'+0, x')}{dx} - \frac{dG(x'-0, x')}{dx} \right] = 1$$

where e.g.

$$\frac{dG(x'+0, x')}{dx} \qquad \text{indicates} \quad \lim_{\epsilon \to 0} \frac{dG(x, x')}{dx} \Bigg|_{x = x'+\epsilon}$$

Now consider the solution away from $x = x'$. For $x \neq x'$, G satisfies the homogeneous equation and the general solution is $A\tilde{y}_1 + B\tilde{y}_2$ where $\tilde{y}_1$, $\tilde{y}_2$ are a pair of lin. indep. solutions of the homogeneous equation. We **choose**

for $x < x'$ G as function of x is $C_1 y_1(x)$,
where $y_1(x)$ is the solution of homogeneous equation that satisfies b.c.s at a

for $x > x'$ G as function of x is $C_2 y_2(x)$,
where $y_2(x)$ is the solution of homogeneous equation that satisfies b.c.s at b

Then $C_1(x')$ and $C_2(x')$ are fixed by the condition that G is continuous at $x = x'$ and G' is discontinuous at $x = x'$

N.B. It is important that the two functions $y_1(x)$, $y_2(x)$ are linearly independent otherwise the method will fail.

Section 7 cont: Green Functions for ODEs

Summary of Method of Constructing a Green Function

1. $\mathcal{L}(x)G(x,x') = \delta(x-x')$. Find general solution of homgeneous equation ($x \neq x'$)
2. Choose $G(x,x')$ as function of x to satisfy boundary conditions of $y(x)$ at a and b
3. Match the two solutions at $x = x'$ by the continuity of G and discontinuity of $1/p(x)$ in $\dfrac{dG}{dx}$ (where $p(x)$ is the coefficient of $\frac{d^2}{dx^2}$ in the $\mathcal{L}$)

Example 1 : Forced Harmonic Oscillator

$$y''(x) + y(x) = f(x) \qquad \text{with b.c.} \quad y(0) = y(\pi/2) = 0$$

1. The Green function satisfies
$$\frac{d^2G(x,x')}{dx^2} + G(x,x') = \delta(x-x') \ ,$$
The solution to the homogeneous equation is $A\sin x + B\cos x$.

2. To satisfy boundary condition $y(0) = 0$
$$y_1(x) = A\sin x \qquad \text{thus} \qquad G(x,x') = C_1(x')\sin x \quad \text{for} \quad x < x'$$
To satisfy boundary condition $y(\pi/2) = 0$
$$y_2(x) = B\cos x \qquad \text{thus} \qquad G(x,x') = C_2(x')\cos x \quad \text{for} \quad x > x'$$

3. The continuity of G and discontinuity of G' ($1/p(x) = 1$ in this example) at $x = x'$ give
$$\begin{aligned} C_2(x')\cos x' - C_1(x')\sin x' &= 0 \\ -C_2(x')\sin x' - C_1(x')\cos x' &= 1 \\ \Rightarrow \qquad & C_1 = -\cos x' \qquad C_2 = -\sin x' \end{aligned}$$

Therefore
$$G(x,x') = -\cos x' \sin x\ \theta(x'-x) - \sin x' \cos x\ \theta(x-x')$$
and the particular solution for the given boundary conditions, but for arbitrary $f(x)$ is

$$y_p(x) = \int_{-\infty}^{\infty} G(x,x')f(x')dx' = -\sin x \int_x^{\pi/2} \cos x' f(x')dx' - \cos x \int_0^x \sin x' f(x')dx'$$

If we were interested in the general solution (without specifying boundary conditions) we could generalise to $y = y_p + Ay_1 + By_2$

N.B. It is important to note that different boundary conditions imply different particular solutions $y_p(x)$ therefore different $G(x,x')$.

Thus specification of the Green function requires both the linear differential operator and suitable boundary conditions. To illustrate the importance of boundary conditions let us again consider the forced harmonic oscillator but this time as an initial value problem. To emphasize the dynamical nature of the problem let us call the independent variable t.

Example 2:

$$y''(t) + y(t) = f(t) \qquad y(0) = 0 \quad y'(0) = 0$$

The forcing begins at $t = 0$ thus $f(t) = 0$ for $t < 0$.

1. The Green function satisfies

   $$\frac{d^2 G(t,t')}{dt^2} + G(t,t') = \delta(t-t') \ ,$$

   The solution to the homogeneous equation is $A \sin t + B \cos t$.

2. To satisfy the boundary conditions $G(0,t') = \frac{d}{dt} G(t,t')\Big|_{t=0} = 0$ we must take $G(t,t') = 0$ for $t < t'$.

   Now construct

   $$G(t,t') = C_1(t') \sin(t) + C_2(t') \cos(t) \qquad t > t' \ .$$

   Since this is an initial value problem there are no boundary conditions at ∞ instead $C_1(t'), C_2(t')$ are fixed by ...

3. ...the continuity of G and discontinuity of G' ($1/p(t) = 1$ in this example) at $t = t'$ giving

   $$\begin{aligned} C_1(t') \sin t' + C_2(t') \cos t &= 0 \\ C_1(x') \cos t' - C_2(t') \sin t' &= 1 \\ \Rightarrow \qquad & C_1(t') = \cos t' \qquad C_2 = -\sin t' \ . \end{aligned}$$

   Thus

   $$G(t,t') = \sin(t-t') \qquad \text{for} \quad t > t'$$

   and the particular solution for the given initial conditions, for arbitrary $f(t)$ is

   $$y_p(t) = \int_{-\infty}^{\infty} G(t,t') f(t') dt' = \int_0^t G(t,t') f(t') dt' = \int_0^t \sin(t-t') f(t') dt' \qquad (11)$$

In the initial value case $G(t,t')$ has a natural interpretation.

- We interpret (11) as

integral of the *stimulus* f at time t' $\times$ the *response* $G(t,t')$ at time t.

- This results from the linearity of the system i.e. we can just integrate up or 'superpose' the responses to stimuli at different times t' to get $y(t)$.

- In the example $G(t, t') = 0$ for $t < t'$ i.e. there is no response in the past (t) to a stimulus in the future (t') — the system is *causal*

$$y_p(t) = \int_{-\infty}^{t} dt' G(t - t') f(t') = \int_{0}^{\infty} du G(u) f(t - u)$$

Similarly in the boundary value problem (example 1) we can interpret

$$y_p(x) = \int_a^b dx' G(x, x') f(x') \tag{12}$$

as the superposition of the responses $G(x, x')f(x')$ at x to the stimulus at x'.

7. 5. Inhomogeneous boundary conditions

Finally let us consider the case of *inhomogeneous* boundary conditions.

$$\alpha y(a) + \beta y'(a) = A \qquad \gamma y(b) + \delta y'(b) = B$$

Let the solution be

$$y(x) = y_p(x) + \tilde{y}$$

where $y_p = \int_a^b dx' G(x, x') f(x')$ satisfies the equation $\mathcal{L} y_p = f(x)$ with homogeneous boundary conditions $\alpha y_p(a) + \beta y_p'(a) = 0$ and $\gamma y_p(b) + \delta y_p'(b) = 0$.

Then $\mathcal{L} y = f(x)$ becomes

$$\begin{aligned} \mathcal{L}(y_p + \tilde{y}) = f(x) + \mathcal{L}\tilde{y} &= f(x) \\ \Rightarrow \mathcal{L}\tilde{y} &= 0 \end{aligned}$$

Similarly the boundary conditions become

$$\alpha \tilde{y}(a) + \beta \tilde{y}'(a) = A \qquad \gamma \tilde{y}(b) + \delta \tilde{y}'(b) = B \ .$$

Thus $\tilde{y}(t)$ is the solution of the homogeneous equation with inhomogeneous boundary conditions.

In the initial value case the same idea works i.e. for inhomogeneous initial conditions

$$y(0) = A \quad , \quad y'(0) = B$$

the solution is $y(t) = y_p(t) + \tilde{y}(t)$ where $y_p(t)$ is the solution of the inhomogeneous equation with homogeneous boundary conditions and $\tilde{y}(t)$ is the solution of the homogeneous equation with inhomogeneous boundary conditions.

Again this is simply a consequence of linearity: $\tilde{y}(t)$ is the solution if there were no forcing term $f(t)$; the solution when $f(t)$ is present is the superposition of $\tilde{y}(t)$ and $y_p = \int_0^t G(t, t') f(t') dt'$, the sum of the reponses to $f(t')$ from all times $t' < t$.

Example 3

$$\frac{d^2}{dt^2}y(t) + y(t) = f(t) \qquad y(0) = A \quad y'(0) = B$$

The solution is, using the result of example 2,

$$y(t) = \int_0^t \sin(t - t')f(t')dt' + \tilde{y}(t)$$

where

$$\tilde{y}(t) = A\cos t + B\sin t\ .$$

7. 6. Change of dependent variable

A second order o.d.e. (4) can be thrown into other standard forms by substituting $y(x) = v(x)u(x)$ which leads (here for $f = 0$) to

$$u'' + \left(\frac{2v'}{v} + p_1\right)u' + \left(\frac{v'' + p_1 v' + p_0 v}{v}\right)u = 0$$

Choosing $v(x)$ to be a known solution recovers (after integration) the result (7).

Another useful choice is

$$v(x) = \exp\left[-\frac{1}{2}\int^x p_1(x')dx'\right] = W^{1/2}(x)$$

which eliminates the u' term and the equation reduces to

$$u''(x) + Q(x)u(x) = 0 \tag{13}$$

where

$$Q(x) = p_0(x) - \frac{1}{2}p_1'(x) - \frac{1}{4}p_1^2(x)\ .$$

(13) is known as the **normal** form or sometimes 'the Schrödinger form'.

Summary of second order linear ODEs:

For second order linear o.d.e.s we need to find one solution y_1 of the homogeneous equation.

Then the second solution y_2 can be find by integrating with the Wronksian (7)

Then a particular solution can be constructed from y_1 and y_2 for any given boundary using the Green function method.