Section 11: Laplace Transforms

11. 1. Laplace Transform as relative of Fourier Transform

For some simple functions the F.T. does not exist e.g. $f(x) = x^2$ since the integral diverges. The divergence at $x = +\infty$ can be remedied by introducing a factor $e^{-x\gamma}$, where $\Re[\gamma]$ is sufficiently large and positive, into the integrand. However we then cause problems at $x = -\infty$. But, for many dynamical problems (where we use t rather than x for the argument) we are only interested in the behaviour for $t > 0$ subject to some initial conditions at $t = 0$.

We can thus consider $f(t)e^{-\gamma t}\theta(t)$ whose F.T. and inverse becomes

$$
\begin{aligned}
g(\omega) &= \int_{-\infty}^{\infty} dt\, f(t)e^{-\gamma t}\theta(t)e^{-i\omega t} = \int_0^{\infty} dt\, f(t)e^{-(\gamma+i\omega)t} \\
f(t)e^{-\gamma t}\theta(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)e^{i\omega t}d\omega
\end{aligned}
$$

Now let $s = \gamma + i\omega$, and rename $g(\omega)$ as $F(s)$, then dropping the theta function (with the understanding that we consider $t \geq 0$) we find

$$
F(s) = \int_0^{\infty} dt\, f(t)e^{-st} \tag{1}
$$

$$
f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\, F(s)e^{st} \tag{2}
$$

(1) is the definition of the Laplace transform often denoted $L[f(t)]$ and (2) is the inversion formula (sometimes known as the Bromwich inversion formula).

In practice (1) will only converge for $\Re[s]$ sufficently large. Consequently, the constant γ in the inversion contour in (2) must be chosen to be *to the right* of any singularity of $F(s)$ in the complex s plane. For $t > 0$ the contour can be closed as a large semicircle to the left (which vanishes by a Jordan's lemma argument)—see figure. One can then use the residue theorem to obtain $f(t)$ from $F(s)$

11. 2. Simple examples

$$
\begin{aligned}
f(t) =& t^\nu \qquad F(s) = \int_0^\infty t^\nu e^{-st} dt = \frac{1}{s^{\nu+1}} \int_0^\infty u^\nu e^{-u} du = \frac{\Gamma(\nu+1)}{s^{\nu+1}} \quad (s>0) \\
f(t) =& e^{\omega t} \qquad F(s) = \frac{1}{s-\omega} \qquad (s>\omega) \\
f(t) =& e^{i\omega t} \qquad F(s) = \frac{1}{s-i\omega} = \frac{s+i\omega}{s^2+\omega^2} \qquad (s>0) \\
f(t) =& \cos\omega t \qquad F(s) = \frac{1}{2(s^2+\omega^2)}[s+i\omega+s-i\omega] = \frac{s}{s^2+\omega^2} \qquad (s>0) \\
f(t) =& \sin\omega t \qquad F(s) = \frac{w}{s^2+\omega^2} \qquad (s>0) \\
f(t) =& \delta(t-a) \qquad F(s) = e^{-as} \qquad (s>0) \\
f(t) =& \theta(t-a) \qquad F(s) = \frac{e^{-as}}{s} \qquad (s>0)
\end{aligned}
$$

Denoting $L[f(t)] = F(s)$ we also have

$$
\begin{aligned}
L[f'(t)] =& \int_0^\infty f'(t) e^{-st} dt = \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty f(t)e^{-st} dt = sF(s) - f(0) \\
L[f''(t)] =& s^2 F(s) - sf(0) - f'(0) \\
L[tf(t)] =& \int_0^\infty t f(t) e^{-st} dt = -\frac{d}{ds} L[f(t)]
\end{aligned}
$$

11. 3. Inversion of Laplace Transform

1. 'Engineering approach' — just look up tables of known Laplace transforms, to work out what is the inverse of a given Laplace Transform
2. Evaluate the inversion integral (2). This has the advantage of our really understanding where the result comes from plus it allows an easy evaluation of the large t behaviour (see later).

Example: $F(s) = 1/s$
We have a simple pole at $s = 0$ thus

$$\text{for} \quad t>0 \qquad f(t) = \quad \text{Residue of } e^{st}/s = 1$$

so $f(t) = \theta(t)$

Example: $F(s) = 1/s^{m+1}$ $\qquad m$ integer
We have pole of order $m+1$ at $s = 0$ thus

$$\text{for} \quad t>0 \qquad f(t) = \text{Residue of } e^{st}/s^{m+1} = \frac{t^m}{m!}$$

Figure 1: Bromwich Inversion integral; γ is to right of any singularity of $F(s)$; for $t > 0$ we close contour to the left

(where to obtain the residue we Taylor expand the exponential) and so $f(t) = \dfrac{t^m}{m!}\theta(t)$

11. 4. Solution of ODEs with initial conditions

As our canonical example we consider

$$\ddot{u}(t) + 2\kappa\dot{u} + \omega_0^2 = 0 \qquad u(0) = u_0 \quad \dot{u}(0) = v_0$$

Take the Laplace transform and use results of 14.2

$$\begin{aligned} s^2F(s) - su(0) - \dot{u}(0) + 2\kappa[sF(s) - u(0)] + \omega_0^2 F(s) &= 0 \\ \Rightarrow F(s) = \frac{(s+2\kappa)u_0 + v_0}{(s+\kappa)^2 + \omega^2} & \qquad \text{where} \quad \omega^2 \equiv \omega_0^2 - \kappa^2 \end{aligned}$$

Now we have to invert $F(s)$.

We can use (i) the engineering approach and look up tables e.g. standard Laplace transforms—which you should check—are

$$L[e^{-\kappa t}\cos\omega t] = \frac{s+\kappa}{(s+\kappa)^2+\omega^2} \qquad L[e^{-\kappa t}\sin\omega t] = \frac{\omega}{(s+\kappa)^2+\omega^2}$$

Or (ii) we can use the inversion integral (2), which we choose to do here.

$$u(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} ds\, \frac{[(s+2\kappa)u_0 + v_0]}{(s-s_+)(s-s_-)} e^{st} \qquad s_\pm = -\kappa \pm i\omega$$

Thus we have two simple poles at $s = s_\pm$ within the contour (for $t > 0$ closed by large semicircle to left) and we can evaluate the sum of the residues

$$\begin{aligned} u(t) &= \frac{[(s_+ + 2\kappa)u_0 + v_0]}{(s_+ - s_-)} e^{s_+ t} + \frac{[(s_- + 2\kappa)u_0 + v_0]}{(s_- - s_+)} e^{s_- t} \\ &= \frac{e^{-\kappa t}}{\omega}\left[(\kappa u_o + v_0)\sin\omega t + \omega u_0 \cos\omega t\right] \end{aligned}$$

11. 5. Convolution

In the context of Laplace transforms we define the convolution as

$$f(t) \circ g(t) = \int_0^t f(t-z)g(z)dz$$

i.e. the limits come from the restriction that the arguments of the two functions be positive.

$$\begin{aligned} L[f(t) \circ g(t)] &= \int_0^\infty dt e^{-st} \int_0^t dz\ f(t-z)g(z) \\ &= \int_0^\infty dz \int_z^\infty dt\ e^{-st} f(t-z)g(z) \\ &= \int_0^\infty dz \int_0^\infty du\ e^{-s(u+z)} f(u)g(z) \qquad \text{where} \quad u = t-z \\ &= L[f]L[g] \end{aligned}$$

One can also show

$$L[f_1 f_2] = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F_1(z) F_2(s-z) dz$$

where F_1, F_2 are Laplace transforms of f_1, f_2 respectively. If $F_1(s)$, $F_2(s)$ exist for $\Re[s] > \alpha_1$, $\Re[s] > \alpha_2$ respectively, we need $\Re[s] - \alpha_2 > \gamma > \alpha_1$.

Example: Forced, damped, harmonic motion

$$\ddot{x} + 2\kappa\dot{x} + \omega_0^2 x = f(t) \qquad x(0) = 0 \quad \dot{x}(0) = 0$$

Then denote L.T. of $x(t)$, $f(t)$ by $\widetilde{x}(s)$, $\widetilde{f}(s)$. We find (see 14.4)

$$\begin{aligned} \widetilde{x}(s) &= \frac{1}{(s+\kappa)^2 + \omega^2} \widetilde{f}(s) \\ \Rightarrow \qquad & x = \int_0^\infty G(t-t') f(t') dt' \\ \text{where} \qquad G(u) &= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} ds \frac{e^{su}}{(s+\kappa)^2 + \omega^2} = \frac{1}{\omega} e^{-\kappa u} \sin \omega u\ \Theta(u) \end{aligned}$$

11. 6. Integral representations

The inversion integral can often be used to develop integral representations of special functions. For example in 14.2 we saw $L[t^\nu] = \dfrac{\Gamma(\nu+1)}{s^{\nu+1}}$. Thus

$$L^{-1}\left[\frac{1}{s^{\nu+1}}\right] = \frac{t^\nu}{\Gamma(\nu+1)}\ .$$

Then using the inversion integral this becomes

$$\frac{t^\nu}{\Gamma(\nu+1)} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\,\frac{e^{st}}{s^{\nu+1}}$$

where $\gamma > 0$ so that we are to the right of the singularity at $s = 0$ which is a branch point for ν noninteger. We take the branch cut along the negative real axis. Then (see figure) when we close the contour to the left we have to avoid the branch cut. Thus a closed contour that encircles no singularities is $C_I + C_{II} + C_{III} + C$ where C_I is the Bromwich contour (parallel to the imaginary axis); C_{II}, C_{III} are quarter circles going from $\gamma + i\infty$ to $-\infty$ and from $-\infty$ to $\gamma - i\infty$; C is a 'loop' or 'Hankel type' contour that comes in from $-\infty$ just above the branch cut, encircles the origin and goes out to $-\infty$ just below the branch cut (similar to the contour we have used for Bessel functions of noninteger order). We may show that the integrals along C_{II} and C_{III} tend to zero —let $s = \gamma + iz$ then the integrals wrt z vanish by a Jordan's Lemma argument. Since the closed contour gives 0 our Bromwich integral is equal to the integral along the contour $-C$.

Figure 2: Closing the inversion contour, and the Hankel type contour C for $F(s) = 1/s^{\nu+1}$

$$\frac{1}{\Gamma(\nu+1)} = t^{-\nu}\frac{1}{2\pi i}\int_C \frac{ds}{s^{\nu+1}}e^{st} = \frac{1}{2\pi i}\int_C dz\, e^z z^{-\nu-1} \qquad (z = st)\ .$$

Thus we have found an integral representation of $\Gamma^{-1}(\nu+1)$. This integral representation can be verified by setting $z = \mathrm{e}^{\mathrm{i}\pi}u$ above and $z = \mathrm{e}^{-\mathrm{i}\pi}u$ below the branch cut. One then identifies the u integral as $\Gamma(-\nu)$ and invokes Euler's reflection formula to get the result (exercise).

11. 7. Coupled equations

To illustrate the power of Laplace transforms let us consider a set of coupled first-order ODE

$$\frac{dN_1}{dt} = -\lambda_1 N_1 \qquad \frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2 \qquad \frac{dN_3}{dt} = \lambda_2 N_2 - \lambda_3 N_3 \tag{3}$$

with initial condition

$$N_1(0) = N \qquad N_2(0) = 0 \qquad N_3(0) = n\ .$$

This system of equation describes a chain of radioactive decay: species 1 decays to species 2 which decays to species 3 which then decays to an inert product. N_i is the number of each species. λ_i are decay rates and are positive. Thus for example the equation for dN_2/dt represents the increase in N_2 due to species 1 decaying and the decrease due to species 2 decaying.

We wish to find $N_i(t)$, in particular the long time behaviour of $N_3(t)$ for example.

To begin we take Laplace transforms of (3) defining $F_1(s)$ as the L.T. of $N_1(t)$ etc:

$$sF_1(s) - N_1(0) = -\lambda_1 F_1(s) \qquad sF_2(s) - N_2(0) = \lambda_1 F_1(s) - \lambda_2 F_2(s)$$

$$sF_3(s) - N_3(0) = \lambda_2 F_2(s) - \lambda_3 F_3(s)$$

which imposing the initial conditions gives

$$F_1(s) = \frac{N}{s+\lambda_1} \qquad F_2(s) = \lambda_1 \frac{F_1(s)}{s+\lambda_2} = \frac{\lambda_1 N}{(s+\lambda_2)(s+\lambda_1)}$$

$$F_3(s) = \frac{n}{s+\lambda_3} + \lambda_2 \frac{F_2(s)}{s+\lambda_3} \quad = \frac{n}{s+\lambda_3} + \frac{\lambda_1\lambda_2 N}{(s+\lambda_3)(s+\lambda_2)(s+\lambda_1)}$$

To find $N_3(t)$ we could in this case invert the expression for $F_3(s)$ by using partial fractions and the inverse transform $L^{-1}\left[1/(s+\lambda_i)\right] = e^{-\lambda_i t}$. However, to get a better understanding of what's going on we use the inversion integral

$$N_3(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} ds\ F_3(s) e^{st}$$

We note that $F_3(s)$ has simple poles at $s = -\lambda_1, -\lambda_2, -\lambda_3$ thus

$N_3 =$ sum of residues

$$= \frac{\lambda_1\lambda_2 N}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_1)} e^{-\lambda_1 t} + \frac{\lambda_1\lambda_2 N}{(\lambda_3-\lambda_2)(\lambda_1-\lambda_2)} e^{-\lambda_2 t} + \frac{\lambda_1\lambda_2 N}{(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)} e^{-\lambda_3 t} + n e^{-\lambda_3 t}$$

Since $\lambda_i > 0$ the long time behaviour is clearly given by λ_i with the smallest real part e.g. if $\lambda_1 < \lambda_2, \lambda_3$

$$\text{for large } t \qquad N_3(t) \sim \frac{\lambda_1\lambda_2 N}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_1)} e^{-\lambda_1 t}\ .$$

Thus the long time behaviour is given by the pole of $F(s)$ that is *furthest to the right* in the complex s plane.

In the present example the intuitive explanation is that the chain of decays is limited by the step with the slowest rate.

Now let us try to generalise (3) by writing it as a matrix equation

$$\frac{d\underline{N}}{dt} = \begin{pmatrix} -\lambda_1 & 0 & 0 \\ \lambda_1 & -\lambda_2 & 0 \\ 0 & +\lambda_2 & -\lambda_3 \end{pmatrix} \underline{N} = M\underline{N} \tag{4}$$

Take the Laplace transform

$$[s\underline{F}(s) - \underline{N}(0)] = M\underline{F}(s) \quad \Rightarrow \quad \underline{F}(s) = (s\mathbb{1} - M)^{-1}\underline{N}(0)$$

where $(s\mathbb{1} - M)^{-1}$ is the inverse matrix of $(s\mathbb{1} - M)$. We can then invert the L.T. by

$$N(t) = \int_{\gamma - i\infty}^{\gamma + i\infty} ds\, (s\mathbb{1} - M)^{-1}\underline{N}(0)e^{st}$$

So in principle we have to find the inverse of $(s\mathbb{1} - M)$.

Now

$$(s\mathbb{1} - M)^{-1} = \frac{\text{Matrix of cofactors}}{\det(s\mathbb{1} - M)} = \frac{\text{Matrix of cofactors}}{\prod_{\mu_i}(s - \mu_i)}$$

The last equality comes from the fact that the determinant of a matrix equals the product of its eigenvalues and the eigenvalues of $s\mathbb{1} - M$ are $s - \mu_i$ where μ_i are the eigenvalues of M.

Thus the singularities of $\underline{F}(s)$ are poles located at $s = \mu_i$ the eigenvalues of the matrix M. Asymptotically (t large) the eigenvalue furthest to the right i.e. with the largest real part will dominate. This is a very general result for equations of the from (4).

N.B. we have assumed the matrix is diagonalisable and we have ignored any degeneracies.

To check the general result we can return to our example (3) where indeed $\det M = -\lambda_1\lambda_2\lambda_3$ and the eigenvalues are $-\lambda_1$, $-\lambda_2$, $-\lambda_3$.

11. 8. Asymptotic behaviour $t \to \infty$

We have seen that if the singularities of $F(s)$ are isolated poles or branch points then the inversion integral reduces to a sum of residues (from the poles) and 'loop integrals' around the branch points (see example of 15.1).

The contribution from a singularity at s_0 comes with a factor $e^{s_0 t}$. Thus for large t the singularity *furthest to the right* in the complex s plane (largest $\Re[s_0]$) dominates

For **poles** it is straightforward to determine the residue (see e.g. 15.2).

For **branch points** one has to do a little more work:

The idea is to expand $F(s)$ around the branch point at s_0:

$$F(s_0) \sim \sum_\nu a_\nu (s - s_0)^{\lambda_\nu}$$

where this is generally an asymptotic expansion (for $s - s_0 \to 0$) and λ_ν may be non-integer

Then a loop integral around the branch point at s_0 will *usually* result in an asymptotic series that can be obtained integrating term by term

$$\frac{1}{2\pi i}\int_{C_{s_0}} ds\, F(s)e^{st} \sim \frac{1}{2\pi i}\int_{C_{s_0}} ds \sum_{\nu} a_\nu (s-s_0)^{\lambda_\nu} e^{st} = e^{s_0 t}\sum_{\nu}\frac{a_\nu}{\Gamma(-\lambda_\nu)t^{\lambda_\nu+1}}$$

Where we have used from 15.1.

$$L^{-1}[s^{-\nu}] = \frac{1}{\Gamma(-\lambda_\nu)t^{\lambda_\nu+1}}$$

An interesting point is that we generate a power series that multiplies the exponential behaviour and when $s_0 = 0$ (which is often the case) the asymptotic behaviour is a power law decay.

Example

$$F(s) = \frac{1}{\sqrt{s(s+a)}} \qquad a > 0$$

This has branch points at $s = 0, -a$ of which $s = 0$ is furthest to the right. Now expand $F(s)$ about $s = 0$:

$$F(s) = \frac{1}{\sqrt{sa}}\frac{1}{(1+s/a)^{1/2}} = \frac{1}{\sqrt{sa}}\left[1 - \frac{s}{2a} + \frac{3s^2}{8a^2} + \cdots\right]$$

In this case the expansion is convergent i.e. it is $s^{-1/2}A(s)$ where $A(s)$ is a Taylor series convergent for $|s| < 1$.

We now inverse transform term by term:

$$f(t) = \frac{1}{\sqrt{a}}\left[\frac{1}{\Gamma(1/2)t^{1/2}} - \frac{1}{2a\Gamma(-1/2)t^{3/2}} + \frac{3}{8a^2\Gamma(-3/2)t^{5/2}} + \cdots\right]$$

Recalling $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(1/2) = \sqrt{\pi}$ we deduce $\Gamma(-1/2) = -2\sqrt{\pi}$, $\Gamma(-3/2) = 4\sqrt{\pi}/3$. Thus we obtain an asymptotic expansion for large t as

$$f(t) = \frac{1}{\sqrt{\pi a}\, t^{1/2}}\left[1 + \frac{1}{4at} + \frac{9}{32a^2t^2} + \cdots\right]$$

11. 9. Other transform pairs

Fourier Transform	$g(k) = \int_{-\infty}^{\infty} dx\, f(x)e^{-ikx}$	$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk g(k)e^{ikx}$
Laplace Transform	$F(s) = \int_0^{\infty} dt\, f(t)e^{-st}$	$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\, F(s)e^{st}$
Mellin Transform	$\phi(z) = \int_0^{\infty} dt\, f(t)t^{z-1}$	$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz\, \phi(z)t^{-z}$
Hilbert Transform	$g(y) = \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} dx\, \frac{f(x)}{x-y}$	$f(x) = \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} dy\, \frac{g(y)}{y-x}$

Where P indicates the Cauchy principal value of the integral. The Mellin transform is particularly useful for asymptotic expansions and is basically a Laplace transform with e^{-t} replaced by t. It generally only exists in a strip e.g. $\alpha < \Re[z] < \beta$.