Section 15: Solution of Partial Differential Equations; Laplace's equation

We consider **Laplace's equation**

$$\nabla^2 u(\underline{x}) = 0$$

or its inhomogeneous version **Poisson's equation**

$$\nabla^2 u(\underline{x}) = \rho(\underline{x}) \ .$$

(To simplify things we have ignored any time dependence in ρ.) The Laplacian is an elliptic operator so we should specify Dirichlet or Neumann conditions on a *closed* boundary S.

15. 1. Homogeneous boundary conditions

By homogeneous boundary conditions we mean $u(\underline{x}) = 0$ when $\underline{x}$ is on S (Dirichlet) or $\partial u(\underline{x})/\partial n = 0$ when $\underline{x}$ is on S (Neumann). In this case we construct a particular solution as usual i.e. the Green function is defined by

$$\nabla^2 G(\underline{x}, \underline{x}') = \delta(\underline{x} - \underline{x}') \tag{1}$$

and we *choose* G to satisfy the same boundary conditions as u. Then

$$u(\underline{x}) = \int_V G(\underline{x}, \underline{x}')\rho(\underline{x}')dV'$$

Example: consider the boundary condition $G(\underline{x}, \underline{x}') \to 0$ as $|\underline{x}| \to \infty$.

We integrate (1) (wrt to $\underline{x}$) over a sphere V centred at $\underline{x}'$ of radius R:

$$\int_V \nabla^2 G(\underline{x}, \underline{x}')dV = 1$$

Now since (1) is isotropic (i.e. no angular dependence) we expect the solution to be of the form $G(\underline{x}, \underline{x}') = G(r)$ where $r = |\underline{x} - \underline{x}'|$. Thus the lhs can be rewritten by using the divergence theorem

$$\int_V \nabla^2 G(\underline{x}, \underline{x}')dV = \int_S (\underline{\nabla} G) \cdot \underline{dS} = 4\pi R^2 \left.\frac{dG}{dr}\right|_{r=R}$$

This yields

$$\frac{dG}{dr} = \frac{1}{4\pi r^2} \qquad \Rightarrow \qquad G = -\frac{1}{4\pi r} + \text{const}$$

The boundary condition that G vanishes at ∞ fixes the constant as zero and we have

$$G(\underline{x}, \underline{x}') = -\frac{1}{4\pi|\underline{x} - \underline{x}'|} \tag{2}$$

This is known as the 'Fundamental solution' of (1). Thus for the form of Poisson's equation used in electrostatics

$$\nabla^2 \phi(\underline{x}) = -\frac{\rho(\underline{x})}{\epsilon_0} \qquad \phi(\underline{x}) = \int \frac{\rho(\underline{x}')}{4\pi\epsilon_0|\underline{x} - \underline{x}'|}\, dV' \ .$$

And we see the Coulomb potential is just the 'Fundamental Green function' for the Laplacian.

15. 2. Inhomogeneous boundary conditions

In this case u or $\partial u/\partial n$ is some nontrivial function prescribed along the boundary. The construction of the particular solution is a little more involved.

We start by noting an identity which is easily verified (see e.g. MP2h)

$$\underline{\nabla} \cdot (\phi \underline{\nabla} \psi - \psi \underline{\nabla} \phi) = \phi \nabla^2 \psi - \psi \nabla^2 \phi$$

then integrating and using the divergence theorem leads to 'Green's identity'

$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, dV = \int_S (\phi \underline{\nabla} \psi - \psi \underline{\nabla} \phi) \cdot \underline{\hat{n}} \, dS$$

We now let $\phi(\underline{x}) = u(\underline{x})$ and $\psi(\underline{x}) = G(\underline{x}', \underline{x})$ and note $\underline{\nabla} u \cdot \underline{\hat{n}} = \dfrac{\partial u}{\partial n}$

$$\int_V \left[u(\underline{x}) \nabla^2 G(\underline{x}', \underline{x}) - G(\underline{x}', \underline{x}) \nabla^2 u(\underline{x}) \right] \, dV = \int_S \left[u(\underline{x}) \frac{\partial G(\underline{x}', \underline{x})}{\partial n} - G(\underline{x}', \underline{x}) \frac{\partial u(\underline{x})}{\partial n} \right] dS$$

The lhs of this equation becomes, assuming $\nabla^2 G(\underline{x}', \underline{x}) = \nabla^2 G(\underline{x}, \underline{x}') = \delta(\underline{x} - \underline{x}')$

$$\text{lhs} \quad = \int_V \left[u(\underline{x}) \delta(\underline{x} - \underline{x}') - G(\underline{x}', \underline{x}) \rho(\underline{x}) \right] \, dV = u(\underline{x}') - \int_V G(\underline{x}', \underline{x}) \rho(\underline{x}) \, dV$$

Finally we interchange $\underline{x}$ and $\underline{x}'$ and rearrange to obtain

$$\boxed{u(\underline{x}) = \int_V G(\underline{x}, \underline{x}') \rho(\underline{x}') \, dV' + \int_S \left[u(\underline{x}') \frac{\partial G(\underline{x}, \underline{x}')}{\partial n'} - G(\underline{x}, \underline{x}') \frac{\partial u(\underline{x}')}{\partial n'} \right] dS' \quad (3)}$$

Notes

1. If we have homogeneous boundary conditions $u = G = 0$ on S or $\dfrac{\partial u}{\partial n} = \dfrac{\partial G}{\partial n} = 0$ on S then the boundary term in the boxed equation vanishes and we are left with the formula for u we used in
2. We have assumed $G(\underline{x}, \underline{x}') = G(\underline{x}', \underline{x})$; generally $G(\underline{x}, \underline{x}') = G^*(\underline{x}', \underline{x})$.
3. We can generalise this Green function solution for finite bondaries to other operators than the Laplacian. In particular the formula will hold for Sturm Liouville operators $\mathcal{L}(\underline{x}) = \underline{\nabla} \cdot \{p(\underline{x}) \underline{\nabla}\} + q(\underline{x})$ with the appropriate boundary conditions. Also one can generalise the formula to the diffusion equation, where for the spatial domain one may have finite boundary conditions, but we omit due to lack of time.

Dirichlet problem

In this case $u = f(\underline{x})$ on S. We *choose*

$$G(\underline{x}, \underline{x}') = 0 \qquad \text{on } S$$

This G is known as the 'Dirichlet Green function'.

Then the boxed equation (3) reduces to

$$u(\underline{x}) = \int_V G(\underline{x}, \underline{x}')\rho(\underline{x}')\, dV' + \int_S f(\underline{x}') \frac{\partial G(\underline{x}, \underline{x}')}{\partial n'}\, dS'$$

The Dirichlet Green function must satisfy both (1) and $G(\underline{x}, \underline{x}') = 0$ for $\underline{x}$ on S. We seek a solution of the form

$$G(\underline{x}, \underline{x}') = F(\underline{x}, \underline{x}') + H(\underline{x}, \underline{x}')$$

where F satisfies (1) and is known as the *fundamental solution*.

$H(\underline{x}, \underline{x}')$ on the other hand satisfies $\nabla^2 H = 0$ and we choose it to enforce $G(\underline{x}, \underline{x}') = 0$ i.e. to cancel F on the boundary.

We have seen that in **3d** $F(\underline{x}, \underline{x}') = -\dfrac{1}{4\pi|\underline{x} - \underline{x}'|}$.

In **2d** one can show (see tutorial) that $F(\underline{x}, \underline{x}') = \dfrac{1}{2\pi} \ln|\underline{x} - \underline{x}'| + \text{constant}$.

Neumann problem

In this case $\dfrac{\partial u}{\partial n} = f(\underline{x})$ is specified on S

We have a consistency condition on f

$$\int_S f(\underline{x})\, dS = \int_S \underline{\nabla} u(\underline{x}) \cdot \underline{\hat{n}}\, dS = \int_V \nabla^2 u(\underline{x})\ dV = \int_V \rho(\underline{x})\, dV$$

It is convenient to choose $\dfrac{\partial G}{\partial n} = \dfrac{1}{A}$ for $\underline{x}$ on S where A is the area of the surface S. With this choice G is known as the 'Neumann Green function'

The boxed equation (3) becomes

$$u(\underline{x}) = \int_V G(\underline{x}, \underline{x}')\rho(\underline{x}')\, dV' + \frac{1}{A}\int_S u(\underline{x}')dS' - \int_S G(\underline{x}, \underline{x}') f(\underline{x}')dS'$$

The second term is like an average of u over the surface.

Again to construct G one writes

$$G(\underline{x}, \underline{x}') = F(\underline{x}, \underline{x}') + H(\underline{x}, \underline{x}')$$

where F is the fundamental solution and H satisfies Laplace's equation inside the volume and is chosen to fix the boundary condition on S.

For both Dirichlet and Neumann problems we end up having to add to our fundamental solution F, a solution H of Laplace's equation chosen to fix the boundary conditions on S.

With the 'method of images' the idea is to choose H itself to be a fundamental solution (or a sum of fundamental solutions) but with their sources *outside* the finite volume V. Let us denote the source for our full Green function as $\underline{x}_0$ i.e. $G(\underline{x}, \underline{x}_0)$. Then

$$\nabla^2 H(\underline{x}, \underline{x}_0) = \sum_{n=1}^{N} q_n \delta(\underline{x} - \underline{x}_n)$$

where the N 'image sources' are of strength q_n and at positions $\underline{x}_n$ outside of V. Thus, inside V, H satisfies Laplace's equation as required. Note that $\underline{x}_0$ only enters rhs implicitly through the choice of q_n, $\underline{x}_n$. Then

$$G(\underline{x}, \underline{x}_0) = F(\underline{x}, \underline{x}_0) + \sum_{n=1}^{N} q_n F(\underline{x}, \underline{x}_n)$$

and we choose q_n $\underline{x}_n$ so that (e.g. for a Dirichlet Green function) $G(\underline{x}, \underline{x}_0) = 0$ on S.

Example: Laplace's equation in 3d for $z > 0$ and $u(\underline{x})$ specified on the $z = 0$ plane.

Here we require the Dirichlet Green function with boundary conditions

$$G(\underline{x}, \underline{x}_0) = 0 \quad \text{on} \quad z = 0 \qquad G(\underline{x}, \underline{x}_0) \to 0 \quad \text{as} \quad |\underline{x}| \to \infty$$

The fundamental solution is

$$F(\underline{x}, \underline{x}_0) = -\frac{1}{4\pi|\underline{x} - \underline{x}_0|}$$

we choose an image source to have strength -1 and to be at the reflection of x_0 in the plane $z = 0$ i.e.

$$\underline{x}_1 = \underline{x}_0 - 2\underline{e}_3(\underline{e}_3 \cdot \underline{x}_0)$$

Thus

$$G(\underline{x}, \underline{x}_0) = -\frac{1}{4\pi|\underline{x} - \underline{x}_0|} + \frac{1}{4\pi|\underline{x} - \underline{x}_1|}$$

and it can be verified that on $z = 0$ the two contributions to G cancel since $|\underline{x} - \underline{x}_0|^2 = (x - x_0)^2 + (y - y_0)^2 + z_0^2$ and $|\underline{x} - \underline{x}_1|^2 = (x - x_0)^2 + (y - y_0)^2 + (-z_0)^2$.

Although we have only illustrated it in the context of Poisson's equation, the method of images is a general technique for constructing the solution of a PDE on a finite region when the boundary surfaces are suitably symmetric (see e.g. tutorial Q9.10).