METHODS OF MATHEMATICAL PHYSICS

Gamma Function; Laplace's Method **Tutorial Sheet 2**

K: key question – explores core material
R: review question – an invitation to consolidate
C: challenge question – going beyond the basic framework of the course
S: standard question – general fitness training!

2.1 **Generalising the gaussian integral formula** [s] Given the formula

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2/2} = \sqrt{\frac{2\pi}{a}}$$

show that:

(i)

$$\int_{-\infty}^{\infty} dx\, e^{-ikx-ax^2/2} = \sqrt{\frac{2\pi}{a}}e^{-k^2/2a}$$

(Hint: try completing the square, then close a contour in the complex plane)

(ii)

$$\int_{-\infty}^{\infty} dx\, e^{iax^2/2} = \sqrt{\frac{2\pi}{a}}e^{i\pi/4}$$

2.2 **Use of Gamma function** [s] Consider the integral

$$\int \mathrm{d}\underline{x}\, e^{-r^2}$$

over all n dimensional space where $\underline{x}$ is the n-dimensional position vector and r is the radial distance from the origin. By evaluating the integral in two ways— i) as a product of n one-dimensional integrals over x_i ii) by using polar coordinates in n dimensions — express the surface area and volume of the n dimensional unit sphere in terms of Gamma functions.

(**N.B.** a circle is 2d sphere, a usual sphere is 3d sphere, and generally you should assume that the area of an n-dimensional sphere is given by $S_n r^{n-1}$)

2.3 **Another use of Gamma function** [s] Show that

$$\int_0^{\infty} e^{-s^p} ds = \frac{\Gamma(1/p)}{p}$$

2.4 **Generalising Laplace's Method** [s] Generalise Laplace's method to calculate the leading approximation to the integrals along the real axis of the form

$$I(x) = \int_a^b f(t)e^{x\phi(t)}\, dt \qquad \text{for} \quad x \gg 0$$

if the near to the stationary point the expansion of ϕ is

$$\phi(t) = \phi(c) + \frac{1}{n!}(t-c)^n\, \phi^{(n)}(c) + \cdots$$

where n is even and $\phi^{(n)}(c) < 0$. You will need to use the result of Q2.3

2.5 **Derivation of Euler's reflection formula** [r] Review the derivation of

$$\Gamma(z)\Gamma(1-z) = \Gamma(1)B(z,1-z) = \int_0^1 dt\; t^{z-1}(1-t)^{-z} = \int_0^\infty dx\; \frac{x^{z-1}}{1+x}$$

where we changed variables $t = x/(1+x)$.

Evaluate the final integration by a contour integral to show

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}\;.$$

Hint: choose the contour to be the same as Hankel's contour.

Why is the resulting expression valid for whole complex plane?

2.6 **Hypergeometric Function** [c] Consider the hypergeometric function defined as

$${}_2F_1(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n}\, \frac{x^n}{n!}$$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$.

Use the Beta function to verify the integral representation

$${}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt\, t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}$$

2.7 **Stirling's Formula** [r] Review the derivation of Stirling's formula by Laplace's method

$$\begin{aligned}
\Gamma(x+1) &= x^{x+1} \int_0^\infty ds\; \exp(x\,[-s+\ln s]) \\
&\simeq x^{x+1} e^{-x} \int_{-\infty}^{\infty} ds\; \exp(-xu^2/2) \\
&= x^{x+1/2} e^{-x} \sqrt{2\pi}
\end{aligned}$$

Now consider calculating the next term in the expansion. Derive the general formula

$$\int_{-\infty}^{\infty} du\, u^n\, e^{-au^2/2} = \begin{cases} 0 & \text{if} \quad n \quad \text{odd} \\ \frac{\sqrt{2\pi}}{a^{(n+1)/2}}(n-1)(n-3)(n-5)....(3)(1) & \text{if} \quad n \quad \text{even} \end{cases} \tag{1}$$

Using this formula work out to which order you have to expand $-s + \ln s$ to calculate the first correction to Stirling's formula and identify the integrals that will contribute.

2.8 **Two integrals related to the Γ function** [s] Compute the two following integrals:

$$\int_0^1 x^x dx$$

and

$$\oint_{\gamma_n} \Gamma(z) dz$$

where γ_n is a circle enclosing $z = -n$ ($n \geq 0$ an integer) anticlockwise and not containing any other negative integers in its interior.

Hint: Attempt to write the first integrand as a series.