METHODS OF MATHEMATICAL PHYSICS

Saddle-point Method **Tutorial Sheet 3**

K: key question – explores core material
R: review question – an invitation to consolidate
C: challenge question – going beyond the basic framework of the course
S: standard question – general fitness training!

3.1 Asymptotic expansion of Hankel function [r]

Using the saddle-pont method evaluate the asymptotic expression of the Hankel function of the *second* kind given by

$$H_\nu^{(2)} = \frac{1}{\pi i}\int_{-\infty-i\epsilon}^{0-i\epsilon} \exp\left[\frac{x}{2}\left(z-\frac{1}{z}\right)\right]\frac{dz}{z^{\nu+1}}$$

where there is a branch cut along the negative real axis.

Ans: $H_\nu^{(2)} \sim \sqrt{\frac{2}{\pi x}}\, e^{-i(x-\nu\pi/2-\pi/4)}$

3.2 Changing variable to obtain form for saddle-point method [s]

Consider

$$I(x) = \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{\exp(-t^2)}{t^{2x}}\, dt$$

where there is a branch cut along the positive real axis.

Change variable to $t = \sqrt{x}\, u$ then use the saddle-point method to calculate the leading behaviour of the integral for large x.

Hint: You should find two saddle points at $\pm i$ but only one will contribute.

Ans: $I(x) \sim \sqrt{\frac{\pi}{2x}}\, x^{-x+1/2} e^{-i\pi x+x}$

3.3 Asymptotic expansion of Legendre Polynomial [s]

The Legendre polynomials $P_n(\cos\alpha)$ may be defined by Schläfli's integral

$$P_n(\cos\alpha) = \frac{1}{2^{n+1}\pi i}\oint \frac{(t^2-1)^n}{(t-\cos\alpha)^{n+1}}\, dt$$

where the contour encircles $t = \cos\alpha$ in an anticlockwise direction. Use the saddle point method to obtain the asymptotic behaviour for large n when $0 < \alpha < \pi$:

(i) Define $f(t) = \ln(t^2-1) - \ln(t-\cos\alpha)$ and show that $f'(t) = 0$ at $t_\pm = e^{\pm i\alpha}$.

(ii) Show that $f''(t_\pm) = \dfrac{\exp \mp i(\alpha+\pi/2)}{\sin\alpha}$.
What is ϕ, the phase of the steepest descent contour through the two saddle-points? Check that the contour goes through the saddle points in the correct sense.

(iii) Use the saddle point formula

$$\int g(z)e^{Nf(z)}\,dz \simeq g(z_0)e^{Nf(z_0)}\,e^{i\phi}\left(\frac{2\pi}{N|f''(z_0)|}\right)^{1/2}$$

to compute the contributions from the two saddle points t_+, t_- as

$$-\frac{e^{in\alpha+3i\pi/4+i\alpha/2}}{\sqrt{2\pi\sin\alpha n}},\;\frac{e^{-in\alpha+i\pi/4-i\alpha/2}}{\sqrt{2\pi\sin\alpha n}}$$

respectively.

(iv) Hence show

$$P_n(\cos\alpha)\sim\sqrt{\frac{2}{\pi\sin\alpha n}}\,\sin(n\alpha+\alpha/2+\pi/4)$$