METHODS OF MATHEMATICAL PHYSICS

Dirac Delta; Green functions for ODE **Tutorial Sheet 4**

K: key question – explores core material
R: review question – an invitation to consolidate
C: challenge question – going beyond the basic framework of the course
S: standard question – general fitness training!

4.1 **Dirac delta function [k]**

(i) Show that $\delta(ax) = \frac{1}{|a|}\delta(x)$

(ii) Show that $\delta(x^2 - a^2) = \frac{1}{2|a|}\left[\delta(x+a) + \delta(x-a)\right]$

(iii) What is $x\delta(x)$? What is $x\,\delta'(x)$?

(iv) Evaluate $\int_{-\infty}^{\infty} dx\, \delta'(x^2 - a^2) e^{iwx}$

(v) For a point $\vec{r_0}$ on the unit sphere show that

$$\delta(\underline{\hat{r}} - \underline{r}_0) = \frac{1}{\sin\theta}\delta(\theta - \theta_0)\delta(\phi - \phi_0)$$

where $\underline{\hat{r}}$ is a unit vector and θ, ϕ are speherical polar coordinates

4.2 **Integral representation of step function [r]**

Show that an integral representation of the Heaviside step function is given by

$$\theta(x) = \frac{1}{2\pi i}\int_{-\infty-i\gamma}^{\infty-i\gamma} \frac{e^{ixk}}{k}\, dk$$

where $\gamma > 0$

4.3 **Wronskian [k]**

(i) Show in the case of a second order ODE

$$y'' + p_1(x)y' + p_0(x)y = 0$$

that the Wronskian of two solutions $Y_1(x), Y_2(x)$

$$W(x) \equiv \begin{vmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{vmatrix}$$

obeys

$$W'(x) = -p_1(x)W(x)\ .$$

(ii)* Show that the Wronskian of n solutions of an n^{th} order ODE obeys

$$W'(x) = -p_{n-1}(x)W(x)\ .$$

4.4 **Green Function [r]**

Consider the equation for driven simple harmonic motion

$$y''(t) + w^2 y(t) = f(t)$$

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(i) Calculate the Wronskian using Abel's formula. Given that one solution of the homogeneous equation is $Y_1(t) = \sin wt$, find a second linearly independent solution.

(ii) Using the Green Function method find a particular solution for the inhomogeneous equation with boundary conditions $y(0) = y(\pi/2w) = 0$.

(iii) Calculate the solution explicitly for $f(t) = k \sin \alpha t$.

4.5 Green function [k]

Construct the Green function for the problem

$$y''(x) = f(x) \quad \text{for} \quad 0 < x < L\ ; \qquad y(0) = 0\ ; \quad \left.\frac{dy}{dx}\right|_{x=L} = 0$$

Write down the solution of the differential equation in terms of integrals over $f(x)$, and verify that your solution satisfies the differential equation and the boundary conditions.

4.6 Another Green function [s]

Construct the Green function for the problem

$$\frac{d}{dx}\left\{x\frac{dy}{dx}\right\} - \frac{n^2}{x}y(x) = f(x) \quad \text{for} \quad 0 < x < 1\ ; \qquad y(0) \quad \text{bounded}\ ; \quad y(1) = 0$$

Treat the cases $n = 0$ and $n \neq 0$ separately. [For $n \neq 0$, the equation is solved by $x^{\pm n}$.]

4.7 Initial value problem [s]

Solve the initial value problem

$$\frac{d^2}{dt^2}y(t) + \alpha\frac{dy(t)}{dt} = f(t) \qquad \text{for} \quad y(0) = A \quad \left.\frac{dy(t)}{dt}\right|_{t=0} = B$$

by constructing the solution in the form

$$y(t) = \int_0^t G(t, t')f(t')dt' + \tilde{y}(t)$$

where the first term satisfies homogeneous initial conditions.