METHODS OF MATHEMATICAL PHYSICS

Fourier Transformations; Solution of ODEs **Tutorial Sheet 6**

K: key question – explores core material
R: review question – an invitation to consolidate
C: challenge question – going beyond the basic framework of the course
S: standard question – general fitness training!

6.1 Fourier Transforms [k]

What is the Fourier transform of

(i) $\delta(x)$ (ii) $\delta'(x)$ (iii) $\exp(-\frac{1}{2}\alpha x^2)$?

6.2 Convolution Theorem [k]

Show that

$$\mathcal{F}[f_1(x)f_2(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} g_1(k')g_2(k-k')dk'$$

where $\mathcal{F}$ denotes Fourier transform and $g_1(k) = \mathcal{F}[f_1]$, $g_2(k) = \mathcal{F}[f_2]$.

6.3 Parseval's Theorem [s]

Consider the Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \exp\frac{in\pi x}{L}\,.$$

Show that

$$C_n = \frac{1}{2L}\int_{-L}^{L} dx\, f(x) \exp\frac{-in\pi x}{L}.$$

Hint: Recall that

$$\delta_{mn} = \frac{1}{2L}\int_{-L}^{L} \exp\frac{i(m-n)\pi x}{L} dx.$$

Now consider the related inverse Fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, g(k)e^{ikx}.$$

Show that

$$\frac{1}{2L}\int_{-L}^{L} dx\, |f(x)|^2 = \sum_{n=-\infty}^{\infty} |C_n|^2 \qquad \text{and} \qquad \int_{-\infty}^{\infty} dx\, |f(x)|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, g(k)g^*(k)$$

6.4 Causal and non-causal systems [s]

Use the Fourier transforms to obtain solutions to the equations

(i) $f''(t) + 2f'(t) + f(t) = g(t)$

(ii) $f'''(t) - 2f'(t) + 4f(t) = g(t)$

in the form of a convolution. Which of these equations describes a causal linear system?

Hint for (ii): one root of $ik^3 + 2ik - 4 = 0$ is $k = 2i$.

6.5 Fourier transform of uniform disk [c]

Consider the function circ($\mathbf{r}$), which is defined as follows,

$$\begin{aligned} \text{circ}(\mathbf{r}) &= 1 \qquad \text{if} \qquad |\mathbf{r}| \leq 1 \\ \text{circ}(\mathbf{r}) &= 0 \qquad \text{otherwise.} \end{aligned}$$

Show that the Fourier transform of $f(\mathbf{r}) = \text{circ}(\mathbf{r})$ in 2 dimensions is

$$\tilde{f}(\mathbf{q}) = 2\pi \frac{J_1(q)}{q}$$

where $q = |\mathbf{q}|$, and J_1 is the Bessel function of index 1.

Hint: The following relation may be useful,

$$\int dr r J_0(r) = r J_1(r)$$

where J_1 is the Bessel function of index 0, which may be written for instance as

$$J_0(r) = \int_0^{2\pi} \frac{d\theta}{2\pi} \cos(r\cos(\theta)) = \int_0^{2\pi} \frac{d\theta}{2\pi} \exp\left[-ir\cos(\theta)\right].$$