METHODS OF MATHEMATICAL PHYSICS

Laplace Transforms **Tutorial Sheet 7**

K: key question – explores core material
R: review question – an invitation to consolidate
C: challenge question – going beyond the basic framework of the course
S: standard question – general fitness training!

7.1 Properties of Laplace transforms [k]

(a) Given that $F(s)$ is the Laplace transform of $f(t)$, such that

$$F(s) \;=\; \mathcal{L}[f(t)] \;=\; \int_0^\infty e^{-st} f(t)dt\,,$$

verify the following basic Laplace transform properties:–

(i) $\mathcal{L}[e^{\alpha t} f(t)] \;=\; F(s-\alpha);$ (ii) $\mathcal{L}[f(t-\alpha)\theta(t-\alpha)] \;=\; e^{-\alpha s}F(s);$

(iii) $\mathcal{L}[f(at)] \;=\; \frac{1}{a}F\Big(\frac{s}{a}\Big);$ (iv) $\mathcal{L}[t^n f(t)] \;=\; \Big(-\frac{d}{ds}\Big)^n F(s).$

(b) Given that the Laplace transform of $\cos t$ is $s/(s^2+1)$, find using the above results the Laplace transforms of

(i) $\cos\omega(t-t_0)\theta(t-t_0);$ (ii) $e^{-\kappa t}\cos\omega t;$ (iii) $t\,\cos\omega t\,.$

7.2 Inverse Laplace transforms [k]

(i) Given that $\mathcal{L}[\sin\omega t] = \omega/(s^2+\omega^2)$, use the convolution theorem to find $\mathcal{L}^{-1}[(s^2+\omega^2)^{-2}]$.
(ii) Given that $\mathcal{L}[(\pi t)^{-1/2}] = s^{-1/2}$, use the convolution theorem to show that

$$\mathcal{L}^{-1}[1/s\sqrt{s+1}] = \mathrm{erf}(\sqrt{t}),$$

where $\mathrm{erf}(x) \equiv \frac{2}{\sqrt{\pi}}\int_0^x du e^{-u^2}$.

7.3 The Browmich inversion formula [s]

Use the Bromwich inversion formula and the residue theorem to obtain the inverse Laplace transforms of the following functions:

(i) $s(s^2+a^2)^{-1}(s^2+b^2)^{-1}$, (ii) $(s^2+\omega^2)^{-2}$, (iii) $e^{-bs}/(s-a)^n$,
(iv)* $s^{-1/2}$.

7.4 Solving ODEs with Laplace transforms

Use the Laplace transform to solve the following initial value problems:

(i) $\ddot{x} - x = t,\quad x(0) = \dot{x}(0) = 0,$

(ii)* $\ddot{x} + \omega^2 x + \alpha^2(x-y) = 0,$
$\ddot{y} + \omega^2 y + \alpha^2(y-x) = 0,$
$x(0) = y(0) = \dot{y}(0) = 0,\quad \dot{x}(0) = u.$

(iii) $x''' + x'' + x' + x + 1 = 0,$
$x(0) = x'(0) = x''(0) = 0.$

7.5 **An integral equation [s]**

Use Laplace transforms to solve the integral equation

$$f(y) = 1 + \int_0^y xe^{-x} f(y-x)\, dx.$$

7.6 **Laguerre Polynomials [s]**

(i) The Laguerre polynomials may be defined as

$$L_n(t) = e^t \frac{d^n}{dt^n}(t^n e^{-t}).$$

Show that $\mathcal{L}[L_n(t)] = n!(s-1)^n/s^{n+1}$. Now consider the Laplace transform of f_n satisfying the equation

$$t\ddot{f}_n + (1-t)\dot{f}_n + nf_n = 0.$$

Deduce that $f_n = L_n$ is a solution to this equation.

(ii) By taking its Laplace transform show that $F(x,t) = e^{-xt/(1-x)}/(1-x)$ is a generating function for the Laguerre polynomials, ie that

$$F(x,t) = \sum_{n=0}^{\infty} \frac{x^n}{n!} L_n(t).$$

7.7 **Riemann Zeta function [s]**

The zeta function is defined for $\Re z > 1$ by $\zeta(z) = \sum_1^\infty s^{-z}$. By considering $\int_0^\infty e^{-st} t^{z-1} dt$ show that

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1} dt.$$

Deduce from this the Hankel representation

$$\zeta(z) = \frac{\Gamma(1-z)}{2\pi i} \int_C \frac{t^{z-1}}{e^{-t}-1} dt,$$

where C is the anticlockwise 'loop' contour around the branch cut

from $-\infty$ to 0 (see lecture 3).

[You will need to recall Euler's reflection formula]

7.8 **Heaviside expansion theorem [s]**

If the Laplace transform $F(s)$ may be written as a ratio

$$F(s) = \frac{g(s)}{h(s)}$$

where $g(s)$ and $h(s)$ are analytic functions, $h(s)$ having simple isolated zeros at $s = s_i$ show that

$$f(t) = L^{-1}\left[\frac{g(s)}{h(s)}\right] = \sum_i \frac{g(s_i)}{h'(s_i)} e^{s_i t}$$