METHODS OF MATHEMATICAL PHYSICS

Solution of Partial Differential Equations **Tutorial Sheet 8**

K: key question – explores core material
R: review question – an invitation to consolidate
C: challenge question – going beyond the basic framework of the course
S: standard question – general fitness training!

8.1 Green function for Helmholtz operator in 1d [s]

Show that the Green function for the modified Helmholtz operator in one dimension, $\mathcal{L}(x) = \frac{d^2}{dx^2} - k^2$, subject to the boundary condition that it should vanish at $x \to \pm\infty$ is

$$G(x, x') = -\frac{1}{2k} e^{-k|x-x'|}$$

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8.2 Inhomogeneous diffusion equation [s]

The density $\rho(x,t)$ of a radioactive gas diffusing into the atmosphere ($x > 0$) from the ground ($x < 0$) satisfies the equation

$$\frac{\partial^2 \rho}{\partial x^2} = \frac{1}{\kappa}\frac{\partial \rho}{\partial t} + \gamma\rho \qquad (\text{for} \quad x > 0) \tag{1}$$

Assuming that $\rho(x, 0) = 0$ and $\rho_x = -\alpha$ at $x = 0$ and that $\rho(x,t) \to 0$ as $x \to \infty$:

(i) Use a Fourier cosine transform to obtain the solution

$$\rho(x,t) = \frac{2\alpha}{\pi} \int_0^\infty \frac{1 - \exp[-\kappa(\gamma + y^2)t]}{\gamma + y^2} \cos(xy) dy$$

(ii) Use a Laplace transform with respect to t to obtain a solution to (1) in the form

$$\rho(x,t) = \int_0^t g(x,\tau) d\tau$$

giving an expression for $g(x,\tau)$

(iii) Demonstrate that the two solutions are equivalent by showing that (i) satisfies

$$\frac{\partial \rho}{\partial t} = g(x,t)$$

8.3* Laplace transform of erfc [c]

(i)* if $L[f(x,t)] = \frac{\exp(-a\sqrt{s})}{s}$ show by using the inversion integral and deforming it into a loop integral around a branch cut, that

$$f(x,t) = \text{erfc}(a/2t^{1/2}) \qquad \text{where} \quad \text{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty du e^{-u^2} .$$

(ii) Conversely, if $f(t) = \mathrm{erfc}(a/2t^{1/2})$, show that $g(t) \equiv f'(t)$ is

$$g(t) = a \exp\left\{-a^2/4t\right\} / 2\sqrt{\pi t^3}\,.$$

**Show that $G(s) \stackrel{\text{def}}{=} L[g(t)] = \exp(-as^{1/2})$, and deduce that $L[f(t)] = \exp(-as^{1/2})/s$. [Hint: calculate $G'(s)$ and thereby show that $G(s)$ satisfies the (simple) differential equation $G'(s) = -a\,G(s)/2s^{1/2}$. A boundary condition for this equation can be found by evaluating $G(0)$ explicitly.]

8.4 Green function for Laplace's equation in 2d [s]

Show that the axially symmetric Green function for Laplace's equation in 2D is

$$G(\underline{x}, \underline{x}') = \frac{1}{2\pi} \ln |\underline{x} - \underline{x}'|$$

8.5 Laplace's equation in the quarter plane[s]

The function $u(x, y)$ satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{for} \quad 0 < x < \infty \quad \text{and} \quad 0 < y < \infty$$

The boundary conditions are

$$u(x, 0) = 0 \qquad\qquad u_x(0, y) = -Q\theta(b - y)$$
$$u_x \text{ and } u_y \to 0 \qquad \text{as} \quad x \text{ or } y \to \infty$$

By taking the Fourier sine transform with respect to y, obtain the solution

$$u(x, y) = \frac{2Q}{\pi} \int_0^\infty \frac{1 - \cos(kb)}{k^2} e^{-kx} \sin(ky)\, dk$$

8.6 The method of images [s]

a) Find the position and magnitude of the image charges appropriate to the following problems:-

(i) A point charge q at distance $d - R > 0$ away from the surface of a conducting sphere of radius R which is earthed ($\phi = 0$ on sphere)

(ii)* The same as part (i) but with the sphere isolated (carries zero net charge)

(iii) A dipole $\underline{p}$ at distance d from a conducting plane, with $\underline{p}$ at an angle θ to the normal to the plane.

b)

(i) For the 1d heat conduction equation determine the Green function.

(ii) For the case of the semi-infinite rod (see 9.5) where the boundary is kept at fixed temperature $T(0, t) = 0$, use the method of images to construct the solution for an initial delta function source at $x = a$, $t = 0$.