15 Bits and pieces

15.1 Casimir effect - forces from nothing

For many quantum systems, such as the harmonic oscillator, there is still some energy associated with the lowest quantum state. This "zero-point" energy is real, and can be measured in the 'Casimir effect'. There is a force between two metallic plates in a vacuum, because moving them would change the wavelength/energy of the zero-point quantised electromagnetic waves between them: this change in energy in response to a move equates to a force.

The wavefunction for transverse standing electromagnetic waves between plates of area A separated by a in the z-direction is:

$$\Phi_n = \exp[i(\boldsymbol{k}.\boldsymbol{r} - \omega_n t)] \sin(k_n z)$$

where $\boldsymbol{k}$ lies in the xy plane and $k_n = n\pi/a$. The energy is $E_n = \hbar\omega_n = hc/\lambda = \hbar c\sqrt{\boldsymbol{k}^2 + k_n^2}$

and the force per unit area is $F = -\frac{dE}{da} = \frac{d}{da}\left(\hbar \int \sum_{n=1}^{\infty} \omega_n\right) dk_x dk_y/(2\pi)^2 = -\frac{\hbar c \pi^2}{240 a^4}$

Solving this involves a trick of multiplying each term by $|\omega_n|^{-s}$, then taking the limit of $s = 0$. This tiny attractive force has now been measured (Bressi, Phys.Rev Letters, 2002)

15.2 What does it mean: Wavefunction collapse and the EPR paradox

The interpretation of collapsing wavefunctions is often regarded as unphysical, or philosophically problematic. There appears to be a contradiction with relativity in the idea that the wavefunction collapses instantaneously throughout space, although the wavefunction is not measurable.

An attractive contrary view to the idea of 'measurement collapsing the wavefunction' is that for a particular system the value of a observable is a property of the particle, and the wavefunction only expresses averages over many particles. This kind of property is known as a hidden variable. As we shall see, this interpretation of quantum mechanics can be tested, and is inconsistent with experimental results.

Consider a two-photon decay from a source (e.g. ^{40}Ca). Two polarisers are oriented along the z-direction, and we detect whether or not the photons pass through the polariser.

The decay is one in which angular momentum is conserved, so the photons must be either both right-polarised ($\mathbf{e}_R$) or both left-polarised ($\mathbf{e}_L$) (they travel in opposite directions). We are dealing with bosons, so the wavefunction can be written as a superposition:

$$|12\rangle = \sqrt{\frac{1}{2}}(\mathbf{e}_{1R}\mathbf{e}_{2R} + \mathbf{e}_{1L}\mathbf{e}_{2L})$$

Now convert into x and y polarisation using $\mathbf{e}_R = (\mathbf{e}_x - i\mathbf{e}_y)$ and $\mathbf{e}_L = (\mathbf{e}_x + i\mathbf{e}_y)$ to give

$$|12\rangle = \sqrt{\frac{1}{2}}(\mathbf{e}_{1x}\mathbf{e}_{2x} + \mathbf{e}_{1y}\mathbf{e}_{2y})$$

From this we can clearly see that the quantum probability of the photon 1 passing through its detector is $\frac{1}{2}$, and if so the wavefunction collapses onto $|12\rangle = \mathbf{e}_{1x}\mathbf{e}_{2x}$ and the conditional probability of the second photon passing through its detector is then 1. Thus quantum mechanics tells us that the probability of both detectors counting is $\frac{1}{2}$.

Contrariwise, a hidden variables argument might say that on production the photons were polarised in a random direction, say θ to the x-axis. In this case the probability of passing through either detector would be $\cos^2\theta$, and the probability of simultaneous counts will be $\langle\cos^4\theta\rangle = 3/8$. The mathematics for particles with correlated spins is similar.

Since the wavefunction collapse and hidden variable approach give different answers, we can do an experiment to see which is correct.

15.3 Hidden Variables: Bell's Inequality and Aspect's experiment

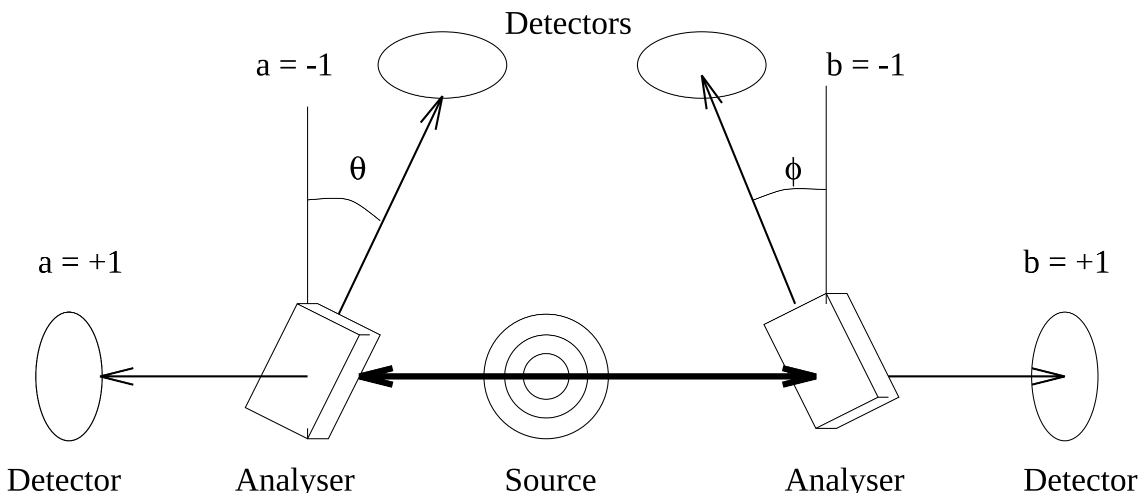

Figure 17: Aspect's Experiment: The polarisations of both photons from the two-photon ^{40}Ca source are measured by analysers at angles of θ and ϕ.

Consider extending the experiment described above to the case of analysers at arbitrary angles which detect all photons. We define measurables a(θ) and b(ϕ) as +1 if the photon is aligned with the analyser and -1 if it is opposed. What, then, is the ensemble average value of $P(\theta,\phi) = \langle a(\theta)b(\phi)\rangle$? Clearly, if a(θ) and b(ϕ) are uncorrelated P=0, but since they come from a common source, this is not the case: their wavefunctions are sometimes referred to as 'entangled'.

If the photons start out with 'hidden variable' polarisation χ, then it is easily shown that:

$$P_{HV}(\theta,\phi) = \frac{1}{2\pi}\oint \left(\cos^2(\theta-\chi) - \sin^2(\theta-\chi)\right)\left(\cos^2(\phi-\chi) - \sin^2(\phi-\chi)\right) d\chi = \frac{1}{2}\cos 2(\theta-\phi)$$

Meanwhile if the wavefunction collapses at the first measurement, taken arbitrarily as A:

$$P_{QM}(\theta,\phi) = \frac{1}{2\pi}\oint \left(\cos^2(\theta-\chi) - \sin^2(\theta-\chi)\right)\left(\cos^2(\theta-\phi) - \sin^2(\theta-\phi)\right) d\chi = \cos 2(\theta-\phi)$$

In 1982, to test this Aspect carried out measurements on ^{40}Ca decays using two different angles for both θ and ϕ. The quantity he evaluated was:

$$S(\theta_1,\phi_1,\theta_2,\phi_2) = P(\theta_1,\phi_1) + P(\theta_2,\phi_2) + P(\theta_2,\phi_1) - P(\theta_1,\phi_2)$$

Where he chose the values which give the largest S: $\theta_1 = \phi_1 + \frac{\pi}{8} = \theta_2 + \frac{2\pi}{8} = \phi_2 + \frac{3\pi}{8}$

The hidden variables theory suggests the result should be S=$\sqrt{2}$, while the wavefunction collapse suggests S=2$\sqrt{2}$ with perfect measurement devices. Imperfections in the measurement will reduce the measured correlation in each case. Aspect measured $S = 2.697\pm0.015$, confirming the quantum prediction.

The apparent complexity of Aspect's experiment is needed to eliminate sources of error due to detector, analyser and source imperfections.

There is an apparent contradiction between quantum mechanics and relativity, in that the *interpretation* of quantum mechanics requires *instantaneous* collapse of the wavefunction. There is no *measurable* quantity for which the two theories give different predictions. "Teleportation" can transport a quantum state arbitrary distances, but it doesn't transfer information instantatneously.

Most of the wavefunctions we have solved are from Schrodingers equation, which treats time and space in different ways. For a properly relativistic approach, they should be equivalent. This discrepancy between quantum and relativity is easily resolved: the Dirac equation provides a fully relativistic wave equation for which the Schrodinger equation is a low energy approximation. A nice thing about the Dirac equation is it can only be solved by spinors: as with quantisation the observed physics turns out to be the only way to solve the mathematics.

The three original papers described in this section are beautifully clear, copies are linked from the course webpage.

15.4 When can things interfere? What counts as a measurement?

Interference from two slits of a single particle with itself remains a difficult concept to understand.

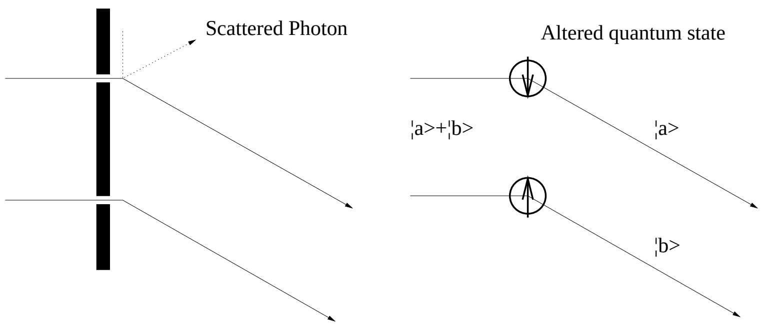

Figure 18: Feynman's 'classical' explanation of the destruction of the interference pattern by measurement, and two separate demonstrations that it is really a quantum effect

Feynman introduced an nice argument based on the uncertainty principle. He argued that the wavelength of light required to detect which way a particle went must be smaller than the slit separation. From the uncertainty principle, it follows that the momentum transfer must be so large that it would destroy the interference pattern. Thus the measurement device destroyed the interference. Unfortunately, more recent experiments show things are more complicated than that.

Eichmann *et al* (Phys.Rev.Lett, 1993) set up a 'two slit' experiment using photon with lead atoms as the scatterers. With careful choice of energy, he was able to arrange that the scattering event changed the internal electronic state of the atom: a process which requires negligible momentum transfer but would allow subsequent measurement of the atomic state and determination which way the particle went. As a consequence, the interference fringes vanish.

Durr *et al* (Nature, 1998) used a standing light wave to scatter rubidium atoms. Added to this was a microwave source which changed the hyperfine state of the atoms at one of the "slits", which could in principle be measured but supplies negligible momentum. The interference pattern disappeared.

Again, quantum mechanics has been shown to give a correct description: non-identical wavefunctions do not interfere even if they describe the same particle! It does not matter whether the measurement of the internal states is actually performed: the mere fact that it could be is enough to destroy the interference.

15.5 Relativistic Quantum Mechanics

The Schroedinger equation itself it clearly inconsistent with relativity; It has second derivatives of space, and first derivatives of time. If we use the relativistic expression for energy $E^2 = |\mathbf{p}|^2 c^2 + m^2 c^4$ we obtain

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \phi(\mathbf{r}, t) = -\hbar^2 c^2 \nabla^2 \phi(\mathbf{r}, t) + m^2 c^4 \phi(\mathbf{r}, t)$$

which is called the Klein Gordon equation. It has solutions describing a relativistic quantum particle, but others which describe particles of negative total energy, together with negative probabilities for finding them! Applied to hydrogen it gets the relativistic kinetic energy correction correct, but it doesn't account for other observed relativistic effects, such as the spin-orbit correction or the Darwin term (see Atomic and molecular physics).

Dirac tried keeping time and space on an equal footing using a linear equation

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = \left\{ c\,\underline{\alpha} \cdot \underline{\hat{p}} + \beta\, mc^2 \right\} \psi(\mathbf{r}, t) = \hat{H}\psi(\mathbf{r}, t) \quad \text{where } \underline{\alpha} \cdot \underline{\hat{p}} = -i\hbar \left(\alpha_x \frac{\partial}{\partial x} + \alpha_y \frac{\partial}{\partial y} + \alpha_z \frac{\partial}{\partial z} \right)$$

Consider a *free particle*, no terms in the Hamiltonian $\hat{H}$ should depend on $\mathbf{r}$ or t as these would describe forces. Dirac assumed that α_i and β are independent of position, time, momentum and energy, so $\underline{\alpha}$ and β commute with $\mathbf{r}$, t, $\underline{\hat{p}}$ and E but not necessarily with each other.

Since relativistic invariance must be maintained, *ie* $E^2 = |\underline{p}|^2 c^2 + m^2 c^4$,

$$\begin{aligned} \hat{H}^2 \psi(\mathbf{r}, t) &= \left(c^2 |\underline{\hat{p}}|^2 + m^2 c^4 \right) \psi(\mathbf{r}, t) \\ &= \left\{ c\,\underline{\alpha} \cdot \underline{\hat{p}} + \beta\, mc^2 \right\} \left\{ c\,\underline{\alpha} \cdot \underline{\hat{p}} + \beta\, mc^2 \right\} \psi(\mathbf{r}, t) \end{aligned}$$

Expand the RHS of this equation, being very careful about the ordering of α_i and β

$$\begin{aligned} &\hat{H}^2\, \Psi(\mathbf{r}, t) \\ &= \left\{ c^2 \left[(\alpha_x)^2 (\hat{p}_x)^2 + (\alpha_y)^2 (\hat{p}_y)^2 + (\alpha_z)^2 (\hat{p}_z)^2 \right] + m^2 c^4 \beta^2 \right\} \psi(\mathbf{r}, t) \\ &\quad + c^2 \Big\{ (\alpha_x \alpha_y + \alpha_y \alpha_x)\, \hat{p}_x \hat{p}_y + (\alpha_y \alpha_z + \alpha_z \alpha_y)\, \hat{p}_y \hat{p}_z + (\alpha_z \alpha_x + \alpha_x \alpha_z)\, \hat{p}_x \hat{p}_z \Big\} \psi(\mathbf{r}, t) \\ &\quad + mc^3 \Big\{ (\alpha_x \beta + \beta \alpha_x)\, \hat{p}_z + (\alpha_y \beta + \beta \alpha_y)\, \hat{p}_z + (\alpha_z \beta + \beta \alpha_z)\, \hat{p}_z \Big\} \psi(\mathbf{r}, t) \end{aligned}$$

relativistic invariance for the free particle requires that the second and third term are zero, and so

$$\begin{aligned} (\alpha_x)^2 = (\alpha_y)^2 = (\alpha_z)^2 = \beta^2 &= 1 \\ \alpha_i\, \alpha_j + \alpha_j\, \alpha_i &= 0 \qquad (i \neq j) \\ \alpha_x\, \beta + \beta\, \alpha_x &= 0 \qquad \text{(and similarly for y, z)} \end{aligned}$$

Thus α_i and β cannot be just numbers. The simplest representation for α and β are 4x4 matrices, meaning that the wavevector is a 4-component vector. When we work this through, there are no negative probabilities, but two of the components turn out to have negative energy. Full details of the derivation are on the course website.

It turns out that the four components accurately describe the two spin states of the electron and the positron. More remarkably, Dirac solved the equation before the positron had even been discovered!