EXACT STEADY STATE PROPERTIES OF THE ONE DIMENSIONAL ASYMMETRIC EXCLUSION MODEL

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Abstract. The asymmetric exclusion model describes a system of particles hopping in a preferred direction with hard core repulsion. Here we review several exact results concerning the steady state of this system which have been obtained recently for periodic and open boundary conditions: density profiles, correlation functions and diffusion constants. We then discuss generalisations to the case of partial asymmetry and to a model with two species of particles.

Key words: Asymmetric exclusion, steady state, diffusion constants, exactly solvable model.

1. Introduction

Models of hopping particles in one dimension [1–5] provide simple but non-trivial realisations of systems out of equilibrium [6–17]. Here we review some recent exact results [18–20] for a family of such models — the asymmetric exclusion process in various geometries and with one or more species of particles. These results, which pertain to steady state properties, have been obtained within a matrix formulation, the description of which will constitute the main part of this presentation.

Let us define the system to be considered. Each site of a one dimensional lattice of N sites is either occupied by one particle or empty. A configuration of the system is characterised by N binary variables $\{\tau_1, \tau_2, \ldots, \tau_N\}$ where $\tau_i = 1$ if site i is occupied by a particle and $\tau_i = 0$ if site i is empty. During an infinitesimal time interval dt, each bond of the lattice has probability dt of being updated. If a bond is updated and there is a particle on the left hand site of the bond, and a hole on the right hand site, the particle will hop across the bond. In other words a particle hops forward with rate 1 whenever there is an empty site on its right.

Different variants of the model can be considered by imposing different boundary conditions for the lattice. For a finite system of N sites two kinds of boundary conditions are often considered:

1. Periodic boundary conditions where $\tau_{i+N} = \tau_i$ and the number of particles $M = \sum_i \tau_i$ is fixed [7, 8, 20].
2. Open boundary conditions, where in time dt a particle may enter the lattice at site 1 with probability αdt (if this site is empty) and a particle at site N may leave the lattice with probability βdt. In this case the number of particles in the system is not conserved [14, 18].

Remark. For finite systems the steady state is unique, that is, the probability $P_t(\tau_i, \cdots, \tau_N)$ of finding the system in configuration $\{\tau_i, \cdots, \tau_N\}$ has a long time limit independent of the initial condition. However as the limits $t \to \infty$ and $N \to \infty$ do not usually commute, the situation for an infinite system is somewhat different and the long time behaviour may depend on initial conditions. For example, suitable initial conditions may produce a shock in the system separating two regions of unequal densities whereas other initial conditions may lead to a homogeneous particle density.

Evolution of the Correlation Functions

Armed with the dynamical rules of the model, one can easily derive the equations which govern the time evolution of any correlation function. For example, if one considers the occupation of site i (for the moment we consider a non-boundary site to avoid choosing any particular boundary conditions) one can write down

$$\tau_i(t+dt) = \begin{cases} \tau_i(t) & \text{with probability } 1-2dt \\ \tau_i(t) + [1-\tau_i(t)]\tau_{i-1}(t) & \text{with probability } dt \\ \tau_i(t)\tau_{i+1}(t) & \text{with probability } dt. \end{cases} \tag{1}$$

The first equation comes from the fact that with probability $1-2dt$, neither of the bonds $i-1, i$ or $i, i+1$ is updated and therefore τ_i remains unchanged. The second equation corresponds to updating bond $i-1, i$: after the update of that bond $\tau_i = 1$ if site i was either occupied before the update or empty but site $i-1$ was occupied. Likewise, the third equation corresponds to updating bond $i, i+1$ after which site i would only be occupied if both site i and site $i+1$ were occupied before the update. If one averages (1) over the events which may occur between t and $t+dt$ and all histories up to time t one obtains

$$\frac{d\langle\tau_i\rangle}{dt} = \langle\tau_{i-1}(1-\tau_i)\rangle - \langle\tau_i(1-\tau_{i+1})\rangle. \tag{2}$$

The same kind of reasoning allows one to write down an equation for the evolution of $\langle\tau_i\tau_{i+1}\rangle$:

$$\frac{d\langle\tau_i\tau_{i+1}\rangle}{dt} = \langle\tau_{i-1}(1-\tau_i)\tau_{i+1}\rangle - \langle\tau_i\tau_{i+1}(1-\tau_{i+2})\rangle. \tag{3}$$

For periodic boundary conditions, where the system has translational invariance, equations of the form (2)–(3) hold for all i. For open boundary conditions one has to consider boundary effects; the equation for the evolution of the one-point correlation function (2) becomes at the boundaries

$$\frac{d\langle\tau_1\rangle}{dt} = \alpha\langle(1-\tau_1)\rangle - \langle\tau_1(1-\tau_2)\rangle, \tag{4}$$

$$\frac{d\langle\tau_N\rangle}{dt} = \langle\tau_{N-1}(1-\tau_N)\rangle - \beta\langle\tau_N\rangle. \tag{5}$$

Once relations of the type (2)–(5) are written, one can in principle calculate the time evolution of any quantity of interest. However, the equation (2) for $\langle\tau_i\rangle$ requires the knowledge of $\langle\tau_i\tau_{i+1}\rangle$ which itself (3) requires the knowledge of $\langle\tau_{i-1}\tau_{i+1}\rangle$ and

$\langle\tau_{i-1}\tau_i\tau_{i+1}\rangle$ so that the problem is intrinsically an N-body problem in the sense that the calculation of any correlation function requires the knowledge of all the others. In what follows, we shall see however that both for periodic and for open boundary conditions, all the correlation functions in the steady state can be calculated exactly.

In the steady state, the correlation functions satisfy equations of the form (2)–(5) where the left hand sides are set to zero.

For the case of *periodic boundary conditions* these equations can, in fact, be solved immediately [6] by recognizing that each configuration (with the correct number M of particles) has equal probability $P_{\rm eq}$:

$$P_{\rm eq} = \left[\begin{pmatrix} N \\ M \end{pmatrix}\right]^{-1}. \tag{6}$$

This can be easily checked by noticing that if all configurations have equal weight, the rate at which the system leaves a given configuration is equal to the number of clusters of particles in that configuration (the first particle of each cluster can hop forward) and the rate at which the the system may enter that configuration is also equal to the number of clusters (by the move of the last particle of each cluster). Then, if one considers for example the two-point correlation functions, it follows that $\langle\tau_i\tau_j\rangle$ will take the same value regardless of the positions of sites i, j. Similarly any n-point correlation will be independent of the positions of the n points (as long as they are all different). With correlation functions of this form it easy to see that the right hand sides of (2)–(3) are automatically zero and similarly any steady state equations for higher order correlation functions would be satisfied.

In the case of *open boundary conditions* one might try to look for a solution of a similar form. However, since the number of particles is not conserved, a corresponding guess as to the form of the stationary probabilities would be that configurations with the same number of particles have the same probability. For $\alpha + \beta = 1$, such a solution does exist (see below) for which all correlation functions are factorised $\langle\tau_i\tau_j\rangle = \langle\tau\rangle^2$ with

$$\alpha = \langle\tau\rangle = 1 - \beta. \tag{7}$$

However, in the general case where $\alpha+\beta \neq 1$, the steady state is non-trivial. The difficulty in calculating the steady state can be seen in (2)–(5): the computation of the one point functions $\langle\tau_i\rangle$ requires the knowledge of the two point functions $\langle\tau_i\tau_{i+1}\rangle$ which in turn require the knowledge of higher correlation functions $\langle\tau_{i-1}\tau_i\tau_{i+1}\rangle$ and $\langle\tau_i\tau_{i+2}\rangle$ and so on. As mentioned earlier, this is a situation quite common in equilibrium statistical mechanics where, although one can write relationships between different correlation functions, there is an infinite hierarchy of equations which in general makes the problem intractable.

In the following we will discuss a way of representing the steady state that for the case of open boundary conditions allows all equal time correlation functions to be computed [18]. A similar approach can also be used in the case of periodic boundary conditions to obtain more complicated steady state properties [20].

2. Matrix Formulation of Steady State for Open Boundaries

Let us now describe a way of calculating the steady state properties in the case of open boundary conditions that we developed in collaboration with V. Hakim and V. Pasquier. This approach had previously been used to solve other problems of statistical mechanics (directed lattice animals and quantum antiferromagnetic spin chains [21, 22, 23]). The idea is to write the weights $f_N(\tau_1, \ldots, \tau_N)$ of the configurations in the steady state as

$$f_N(\tau_1, \ldots, \tau_N) = \langle W| \prod_{i=1}^{N} [\tau_i D + (1-\tau_i)E]\, |V\rangle, \tag{8}$$

where D, E are matrices, $\langle W|, |V\rangle$ are vectors (we use the standard Bra Ket notation of quantum mechanics) and τ_i are the occupation variables. In other words in the product (8) we use matrix D whenever $\tau_i = 1$ and E whenever $\tau_i = 0$. In general, since the matrices D and E do not commute, the weights $f_N(\tau_1, \ldots, \tau_N)$ are complicated functions of the configuration $\{\tau_1, \ldots, \tau_N\}$. As the weights $f_N(\tau_1, \ldots, \tau_N)$ given by (8) are usually not normalised, the probability $p_N(\tau_1, \ldots, \tau_N)$ of a configuration $\{\tau_1, \ldots, \tau_N\}$ in the steady state is

$$p_N(\tau_1, \ldots, \tau_N) = f_N(\tau_1, \ldots, \tau_N) \left[\sum_{\tau_1=1,0} \cdots \sum_{\tau_N=1,0} f_N(\tau_1, \ldots, \tau_N) \right]^{-1}. \tag{9}$$

Of course, from looking at (8) it is not obvious that such matrices D, E and vectors $\langle W|$, $|V\rangle$ exist. We shall see, however, that it is possible to choose these matrices and vectors so that $f_N(\tau_1, \ldots, \tau_N)$ given by (8) are indeed the actual weights in the steady state.

Before presenting some explicit forms for the matrices and vectors involved in (8) let us show how the approach leads to a straightforward computation for the correlation functions. If one defines the matrix C by

$$C = D + E, \tag{10}$$

it is clear that $\langle \tau_i \rangle_N$ defined by

$$\langle \tau_i \rangle_N = \sum_{\tau_1=1,0} \cdots \sum_{\tau_N=1,0} \tau_i f_N(\tau_1, \ldots, \tau_N) \left[\sum_{\tau_1=1,0} \cdots \sum_{\tau_N=1,0} f_N(\tau_1, \ldots, \tau_N) \right]^{-1}, \tag{11}$$

can be calculated through the following formula

$$\langle \tau_i \rangle_N = \frac{\langle W|C^{i-1} D C^{N-i}|V\rangle}{\langle W|C^N|V\rangle}. \tag{12}$$

In the same way, any higher correlation will take a simple form in terms of these matrices. For example, when $i < j$, $\langle \tau_i \tau_j \rangle_N$ is equal to

$$\langle \tau_i \tau_j \rangle_N = \frac{\langle W|C^{i-1} D C^{j-i-1} D C^{N-j}|V\rangle}{\langle W|C^N|V\rangle}. \tag{13}$$

Therefore, all we require in order to be able to calculate arbitrary spatial correlation functions is that the matrix elements of any power of $C = D + E$ have manageable expressions.

One can show [18] that if the matrices D, E and the vectors $\langle W|$, $|V\rangle$ satisfy (14)–(16):

$$D|V\rangle = \frac{1}{\beta}|V\rangle, \tag{14}$$

$$DE = D + E, \tag{15}$$

$$\langle W|E = \frac{1}{\alpha}\langle W|, \tag{16}$$

then (8) does give the steady state.

We shall not repeat here the proof that (14)–(16) are sufficient conditions to give the weights in the steady state. It is however easy to check that the relations (2)–(5) will be satisfied in the steady state provided that the corresponding identities hold:

$$D\ E\ (D+E) = (D+E)\ D\ E, \tag{17}$$

$$D\ E\ D\ (D+E) = (D+E)\ D\ D\ E, \tag{18}$$

$$\alpha\ \langle W|\ E\ (D+E) = \langle W|\ D\ E, \tag{19}$$

$$D\ E\ |V\rangle = \beta\ (D+E)\ D\ |V\rangle, \tag{20}$$

and that these relations are immediate consequences of the algebraic rules (14)–(16). Another easy check that (14)–(16) do give the right steady state is to look at some special configurations. If one takes the case of a configuration where the first p sites are empty and the last $N-p$ are occupied, it is easy to show that in the steady state one must have

$$\langle W|E^{p-1}DED^{N-p-1}|V\rangle = \alpha\langle W|E^pD^{N-p}|V\rangle + \beta\langle W|E^pD^{N-p}|V\rangle \tag{21}$$

since this expresses that during a time interval dt the probability of entering and leaving the configuration are the same. Here again, this equality appears as a very simple consequence of the algebraic rule (14)–(16).

For the line $(\alpha + \beta = 1)$ we mentioned above that the steady state becomes trivial. This is reflected by the fact that one can choose commuting matrices D and E to solve (14)–(16). If D and E commute one can write

$$\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\langle W|V\rangle = \langle W|D+E|V\rangle = \langle W|DE|V\rangle = \langle W|ED|V\rangle = \frac{1}{\alpha\beta}\langle W|V\rangle. \tag{22}$$

As $\langle W|V\rangle \neq 0$, this clearly implies that $\alpha + \beta = 1$. This is a well known special case where the steady state is factorised ($f_N(\tau_1, \ldots, \tau_N)$ depends only on $\sum_i \tau_i$ and all connected correlations vanish). Under this condition $(\alpha + \beta = 1)$, one can choose the matrices D and E to be uni-dimensional, with $D = \beta^{-1}$ and $E = \alpha^{-1}$.

The previous remark also shows that for $\alpha + \beta \neq 1$, the size of the matrices D, E must be greater than one. The next question is whether one can find finite dimensional matrices that will satisfy (14)–(16). It turns out that one can prove [18] that this is impossible (if D and E were finite dimensional matrices, the relation

$DE = D + E$ would imply that $D = E(1-E)^{-1}$ which itself would imply that the matrices D and E commute). So the only possibility left is to use infinite dimensional matrices.

In order to perform calculations within the matrix formulation there are basically two approaches one can take. Either one can work with the algebra (14)–(16) directly, or one can make a particular choice of matrices and use it to the full. In the latter case there are several possible choices for the matrices D, E and vectors $\langle W|$, $|V\rangle$ that satisfy (14)–(16). One particularly simple choice, which has proved useful in the extensions of the approach to be discussed below is

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}, \tag{23}$$

$$\langle W| = \left(1,\ \left(\tfrac{1-\alpha}{\alpha}\right),\ \left(\tfrac{1-\alpha}{\alpha}\right)^2,\ \dots\right), \qquad |V\rangle = \begin{pmatrix} 1 \\ \left(\frac{1-\beta}{\beta}\right) \\ \left(\frac{1-\beta}{\beta}\right)^2 \\ \vdots \end{pmatrix}. \tag{24}$$

This choice makes the particle-hole symmetry of the problem apparent since the matrices D and E have very similar forms and the boundary conditions α and β only appear in the vectors $\langle W|$ and $|V\rangle$. For this choice (23) of D, E the elements of C^N (where $C = D + E$ and N denotes the Nth power of matrix C) are given by

$$\left(C^N\right)_{nm} = \binom{2N}{N+n-m} - \binom{2N}{N+n+m}. \tag{25}$$

Expression (25) can be obtained by noting that $\left(C^N\right)_{nm}$ is proportional to the probability that a random walker who starts at site $2m$ of a semi-infinite chain with absorbing boundary at the origin, is at site $2n$ after $2N$ steps of a random walk. This probability may be calculated by the method of images.

An apparent disadvantage of this choice (23)–(24) is that, due to the form of $\langle W|$ and $|V\rangle$, one has to sum geometric series to obtain the correlation functions and these series diverge in some range of α, β (in fact $\alpha + \beta \le 1$). However, at least for finite N, all expressions are rational functions of α, β so that in principle one can obtain results for $\alpha + \beta \le 1$ by analytic continuation from those for $\alpha + \beta > 1$.

Other choices of matrices and vectors are possible [18], which solve the equations (14)–(16). For example, a possible choice of D, E, $\langle W|$, $|V\rangle$, that avoids the divergences is

$$\tilde{D} = \begin{pmatrix} 1/\beta & a & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}, \qquad \tilde{E} = \begin{pmatrix} 1/\alpha & 0 & 0 & 0 & \dots \\ a & 1 & 0 & 0 & \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}, \tag{26}$$

$$\langle \tilde{W}| = (1, 0, 0, \ldots), \qquad |\tilde{V}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \tag{27}$$

where

$$a^2 = \frac{\alpha + \beta - 1}{\alpha\beta}. \tag{28}$$

The fact that a^2 may be negative is of no importance, because in the calculation of any required matrix element a only enters through a^2. One should note that for $\alpha = \beta = 1$, we have $a = 1$ and (26)–(27) coincides with our previous bidiagonal choice (23)–(24). Also, a vanishes for $\alpha + \beta = 1$ so that the $1, 1$ elements of the matrices $\tilde{D}, \tilde{E}$ decouple from the other elements. This choice of matrices then becomes, for the purposes of our calculations, one-dimensional as is sufficient for this special case of α and β.

Instead of using explicit forms for the matrices, one can calculate directly matrix elements such as those which appear in (12)–(13) from the commutation rules (14)–(16). For example, one can easily show that

$$\frac{\langle W|C|V\rangle}{\langle W||V\rangle} = \frac{\langle W|D + E|V\rangle}{\langle W||V\rangle} = \frac{1}{\alpha} + \frac{1}{\beta}, \tag{29}$$

$$\frac{\langle W|C^2|V\rangle}{\langle W||V\rangle} = \frac{\langle W|D^2 + ED + E^2 + D + E|V\rangle}{\langle W||V\rangle} = \frac{1}{\alpha^2} + \frac{1}{\alpha\beta} + \frac{1}{\beta^2} + \frac{1}{\alpha} + \frac{1}{\beta}. \tag{30}$$

The general expression of $\langle W|C^N|V\rangle$ (where $C = D + E$) for all values of α and β has been shown to be [18]

$$\frac{\langle W|C^N|V\rangle}{\langle W||V\rangle} = \sum_{p=0}^{N} \frac{p\ (2N - 1 - p)!}{N!\ (N - p)!}\ \frac{\beta^{-p-1} - \alpha^{-p-1}}{\beta^{-1} - \alpha^{-1}}. \tag{31}$$

Some Results

Once the matrix elements of C are known, expressions for several quantities can be derived. For example, in the steady state, the current through the bond $i, i+1$ is simply $J = \langle \tau_i(1 - \tau_{i+1})\rangle$, because during a time dt, the probability that a particle jumps from i to $i + 1$ is $\tau_i(1 - \tau_{i+1})dt$. Therefore, J is given by

$$J = \frac{\langle W|C^{i-1}DEC^{N-i-1}|V\rangle}{\langle W|C^N|V\rangle} = \frac{\langle W|C^{N-1}|V\rangle}{\langle W|C^N|V\rangle}, \tag{32}$$

where we have used the fact (15) that $DE = C$. This expression is independent of i, as expected in the steady state. From the large N behaviour of the matrix elements $\langle W|C^N|V\rangle$ given by (31) one can show [18] that there are three different phases where the current J is given by

$$J = \begin{cases} \frac{1}{4} & \text{if } \alpha \geq \frac{1}{2} \text{ and } \beta \geq \frac{1}{2}, \\ \\ \alpha(1-\alpha) & \text{if } \alpha < \frac{1}{2} \text{ and } \beta > \alpha, \\ \\ \beta(1-\beta) & \text{if } \beta < \frac{1}{2} \text{ and } \alpha > \beta. \end{cases} \tag{33}$$

Thus, the phase diagram consists of three phases: $\alpha > \frac{1}{2}$, $\beta > \frac{1}{2}$; $\alpha < \frac{1}{2}$, $\beta > \alpha$; $\beta < \frac{1}{2}$, $\alpha > \beta$. This is exactly the phase diagram predicted by the mean field theory [9, 14, 17].

From the knowledge of the matrix elements $\langle W|C^N|V\rangle$, one can also obtain [18] exact expressions for all equal time correlation functions. For example the profile $\langle \tau_i \rangle_N$ is given by

$$\begin{aligned} \langle \tau_i \rangle_N \ &= \sum_{p=0}^{n-1} \frac{2p!}{p!\ (p+1)!} \frac{\langle W|C^{N-1-p}|V\rangle}{\langle W|C^N|V\rangle} \\ &\quad + \frac{\langle W|C^{i-1}|V\rangle}{\langle W|C^N|V\rangle} \sum_{p=2}^{n+1} \frac{(p-1)(2n-p)!}{n!\ (n+1-p)!} \beta^{-p} \end{aligned} \tag{34}$$

where $n = N - i$. Several limiting behaviours (N large, i large) are discussed in [18]. In the case $\alpha = \beta = 1$, one can even perform the sums in (34) to obtain [14]

$$\langle \tau_i \rangle_N = \frac{1}{2} \ + \ \frac{N - 2i + 1}{4} \frac{(2i)!}{(i!)^2} \frac{(N!)^2}{(2N+1)!} \frac{(2N-2i+2)!}{[(N-i+1)!]^2}. \tag{35}$$

3. Diffusion Constant and Non-Equal Time Correlation Functions for Periodic Boundary Conditions

One can also try to extend the matrix approach to calculate more general steady state properties than equal time correlation functions. The first result of this kind [20] is the exact expression of the diffusion constant Δ for a system of M particles on a ring of N sites in the fully asymmetric case (each particle jumps to its right neighbour with probability dt when the right neighbour is empty). If we consider a tagged particle (which has exactly the same dynamics as the $M-1$ other particles) and if we call Y_t the number of hops performed by this tagged particle between time 0 and time t, one expects that in the long time limit:

$$\lim_{t\to\infty} \frac{\langle Y_t \rangle}{t} = v; \ \lim_{t\to\infty} \frac{\langle Y_t^2 \rangle - \langle Y_t \rangle^2}{t} = \Delta. \tag{36}$$

The velocity v and the diffusion constant Δ are given by

$$v = \frac{N-M}{N-1}; \quad \Delta = \frac{(2N-3)!}{(2M-1)!\ (2N-2M-1)!} \left[\frac{(M-1)!\ (N-M)!}{(N-1)!} \right]^2. \tag{37}$$

A derivation of this result based on the matrix ideas described above is given in [20]. Let us discuss here its connection with non-equal time correlation functions of the τ_i variables. In order to see this, it is convenient to introduce another random variable $\widetilde{Y}_t$ which represents the number of particles which have jumped from site 1 to site 2 between time 0 and time t. Since the particles cannot overtake each other it is clear that

$$\lim_{t\to\infty}\frac{\langle Y_t\rangle}{t}=\frac{N}{M}\lim_{t\to\infty}\frac{\langle\widetilde{Y}_t\rangle}{t};\quad \lim_{t\to\infty}\frac{\langle Y_t^2\rangle-\langle Y_t\rangle^2}{t}=\frac{N^2}{M^2}\lim_{t\to\infty}\frac{\langle\widetilde{Y}_t^2\rangle-\langle\widetilde{Y}_t\rangle^2}{t}. \qquad (38)$$

It is then rather easy to see how the moments of $\widetilde{Y}_t$ are related to unequal time correlation functions. If one decomposes the time t into T infinitesimal time intervals dt with $T=t/dt$, one can write $\widetilde{Y}_t$ as

$$\widetilde{Y}_t=\sum_{k=1}^{T}a_k \qquad (39)$$

where $a_k=1$ if a particle jumps from site 1 to site 2 at time kdt and $a_k=0$ otherwise. All the a_k are random variables with

$$a_k=\begin{cases}1 \text{ with probability } \tau_1(kdt)(1-\tau_2(kdt))dt\\ \\ 0 \text{ with probability } 1-\tau_1(kdt)(1-\tau_2(kdt))dt;\end{cases} \qquad (40)$$

then

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\left[\langle\widetilde{Y}_t^2\rangle-\langle\widetilde{Y}_t\rangle^2\right]\\ &\quad=\lim_{t\to\infty}\left(\frac{1}{t}\sum_{k=1}^{T}\left[\langle a_k^2\rangle-\langle a_k\rangle^2\right]+\frac{2}{t}\sum_{k=1}^{T}\sum_{k'=k+1}^{T}\left[\langle a_k a_{k'}\rangle-\langle a_k\rangle\langle a_{k'}\rangle\right]\right). \qquad (41)\end{aligned}$$

Taking the continuous time limit ($dt\to 0$) one obtains for $t\to\infty$

$$\begin{aligned}\frac{M^2}{N^2}\Delta \;=\;& \langle\tau_1(1-\tau_2)\rangle\\ &+2\int_0^\infty dt\left[\langle\tau_1(t)(1-\tau_2(t))\tau_1(0)(1-\tau_2(0))\rangle-\langle\tau_1(1-\tau_2)\rangle^2\right]. \qquad (42)\end{aligned}$$

So we see that the exact expression of Δ gives some information about unequal time correlation functions. Of course it would be interesting to know whether the matrix approach could be sufficiently refined to give exact expressions for all unequal time correlation functions; however at present this seems to us a very remote goal.

Two limiting cases of (37) are worth mentioning. First if one takes the limit $N\to\infty$ keeping M fixed, one finds

$$\Delta=\frac{[(M-1)!]^2 4^{M-1}}{(2M-1)!}. \qquad (43)$$

In that limit, it is clear that the particles almost never see each other and one might fancy that $\Delta=1$, the value it takes when there is a single particle in the system.

However this is not the case and Δ depends on M because the 'collisions' between two particles are highly correlated in time.

Another limit one can consider is that of a given density ρ of particles in an infinite system ($M = N\rho$ as $N \to \infty$ in (37))

$$\Delta \simeq \frac{\sqrt{\pi}}{2} \left[\frac{(1-\rho)^{3/2}}{\rho^{1/2}}\right] \frac{1}{N^{1/2}}. \tag{44}$$

The fact that Δ vanishes as $N \to \infty$ indicates that in the infinite system the fluctuations of the tagged particle are subdiffusive. This can be seen by considering that, for finite t and N, the quantity $\langle Y_t^2 \rangle - \langle Y_t \rangle^2$ is a function of the two variables t and N. When both t and N are large, one can expect the following sort of scaling form:

$$\langle Y_t^2 \rangle - \langle Y_t \rangle^2 \simeq t^{2\omega} g(t/N^{\gamma}). \tag{45}$$

When $t \to \infty$ first and N is large one knows from the above results that

$$\langle Y_t^2 \rangle - \langle Y_t \rangle^2 \sim t\ N^{-1/2}. \tag{46}$$

This of course gives some constraints on the exponents ω and γ and on the behaviour of the function g for large values of its argument:

$$g(z) \sim z^{1-2\omega} \qquad \text{as} \qquad z \to \infty \tag{47}$$

with $\gamma(1-2\omega) = \frac{1}{2}$. To determine the values of the exponents ω and γ one needs another relation which can be obtained via the following additional argument: for large N, one can ask at what time t does the tagged particle notice that it moves on a finite lattice of size N instead of an infinite lattice. To estimate this time one can use the result [7, 8] that the longest relaxation time in the system scales like $N^{3/2}$. Therefore, $\gamma = \frac{3}{2}$ and one obtains $\omega = \frac{1}{3}$.

In the hope of being able to calculate more general time correlations in the steady state, one can wonder whether result (37) can be generalised. The simplest extension one can consider is the case of open boundary conditions. In that case if one denotes by Y_t the number of particles which have entered the lattice at site 1 between times 0 and t one can evaluate the current and diffusion constant

$$\lim_{t\to\infty} \frac{\langle Y_t \rangle}{t} = J, \quad \lim_{t\to\infty} \frac{\langle Y_t^2 \rangle - \langle Y_t \rangle^2}{t} = \Delta, \tag{48}$$

by solving the problem exactly on the computer for all system sizes $1 \leq N \leq 10$. It seems very likely that Δ is given by following expression in the case $\alpha = \beta = 1$

$$J = \frac{N+2}{4N+2}, \quad \Delta = \frac{3\ (4N+1)!\ N!\ (N+1)!\ [(N+2)!]^2}{2\ [(2N+1)!]^3\ (2N+3)!}. \tag{49}$$

(The expression for J corresponds of course to that calculated in Section 2.)

We have started to develop a matrix approach to establish this result but the work is yet to be completed and at present we do not know whether the matrix approach can be adapted to prove this result and (49) should be considered as a conjecture.

4. More than One Species of Particle

One possible generalisation of the model is to the case of more than one species of particles. For example one can consider a system containing two species of particles, which we represent by 1 and 2, and holes represented by 0 in which the hopping rates of the two species of particles are

$$\begin{array}{l} 1\ 0 \rightarrow 0\ 1 \text{ with rate } 1, \\ 2\ 0 \rightarrow 0\ 2 \text{ with rate } \gamma, \\ 1\ 2 \rightarrow 2\ 1 \text{ with rate } \delta. \end{array}$$

Even for the case of periodic boundary conditions the steady state of this model is in general non-trivial. Nevertheless, the steady state weights may be obtained by writing them in the form [19]

$$\mathrm{trace}(X_1 X_2 \cdots X_N) \tag{50}$$

where $X_i = D$ if site i is occupied by a 1 particle, $X_i = A$ if it is occupied by a 2 particle and $X_i = E$ if it is empty. The translational invariance of a product of matrices under the trace operation used in (50), reflects the translational invariance of the periodic boundary conditions. One can prove that (50) gives the steady state of this system provided that the matrices D, A and E satisfy the following algebra:

$$DE = D + E, \quad \delta DA = A, \quad \gamma AE = A. \tag{51}$$

The second two of these equations are satisfied when A is given by

$$A = |V\rangle\langle W| \tag{52}$$

and

$$D|V\rangle = \frac{1}{\delta}|V\rangle, \quad \langle W|E = \frac{1}{\gamma}\langle W|. \tag{53}$$

So one can use any of the matrices D, E presented for the case of open boundary conditions ((23) and (26)) and construct matrix A from the vectors $\langle W|$, $|V\rangle$ ((24) and (27)) with α replaced by γ and β replaced by δ.

A case of the two species problem of particular interest is that of first and second class particles. This corresponds to $\gamma = \delta = 1$ so that all hopping rates are 1 and both first and second class particles hop forward when they have a hole to their right, but when a first class particle has a second class particle to its right the two particles interchange positions. In regions of a low density of first class particles and a high density of holes, a second class particle will tend to move forward whereas in a high density of first class particles and a low density of holes a second class particle will tend to move backward. For this reason, second class particles were first introduced in the context of an infinite system in order to track the position of shocks (recall that a shock is a change in the density of particles over a microscopic distance) [24, 25, 26]. On a finite system with periodic boundary conditions the steady state is unique and corresponds to a uniform density. However it has been shown that even in the case of periodic boundary conditions one can use a finite density of second class particles to probe the structure of shocks [26]. The idea is that from the point of view of any particular second class particle, those second class particles to its right are equivalent to first class particles whereas second class particles to its left are equivalent to holes. Thus, by calculating the density profile of first and second class particles in a finite system as seen from a second class particle located at the origin, one can construct shock profiles by taking the limit of an infinite system and using the density of first class particles to the left of the second class particle as the profile to the left of the origin and the density of first and second class particles to the right of the second class particle as the profile to the right of the origin [19].

An interesting result concerns the case of a finite number of second class particles in an infinite uniform system of first class particles at density ρ. It can be shown that they form an algebraic bound state, i.e., the probability of finding them a distance r apart decays like a power law in r. For example, in the case of two second class particles in an infinite system of first class particles at density ρ, the probability $P(r)$ of finding them a distance r apart is given by [19]

$$P(r) = \rho(1-\rho) \sum_{p=0}^{r-1} \rho^{2p} \; (1-\rho)^{2r-2p-2} \frac{r!(r-1)!}{p!(p+1)!(r-p)!(r-p-1)!} \tag{54}$$

which decays for large r as

$$P(r) = \frac{1}{2\sqrt{\pi\rho(1-\rho)}} \; \frac{1}{r^{3/2}}. \tag{55}$$

Thus, the two second class particles form a bound state although their average distance is infinite.

Using the matrix approach one can also calculate [27] a diffusion constant Δ for a single second class particle in the presence of M first class particles by considering Y_t as the distance forward (the number of hops forward minus the number of hops backwards) travelled by the second class particle between time 0 and time t, and define a diffusion constant Δ through (36). One finds

$$\begin{aligned}\Delta \;=\; 2 \; \frac{(2N-3)!}{(2M+1)!(2N-2M-1)!} \; \left[\frac{M!(N-M-1)!}{(N-1)!}\right]^2 \\ \times \left[(N-5)M(N-M-1) + (N-1)(2N-1)\right].\end{aligned} \tag{56}$$

(Here the velocity of the second class particle is $v = (N-2M-1)/(N-1)$). The formula simplifies when the $N \to \infty$ limit is taken with $M = N\rho$ and the leading order of (56) is

$$\Delta \simeq \tfrac{1}{4}(N\pi\rho(1-\rho))^{1/2}. \tag{57}$$

This large N dependence contrasts with that of the equivalent formula (44) for the diffusion constant of a first class particle which behaves as $N^{-1/2}$. It is consistent with the idea [28] that in an infinite system a single second class particle displays superdiffusive fluctuations in its position ($\langle Y_t^2\rangle - \langle Y_t\rangle^2 \sim t^{4/3}$).

5. Conclusion

The matrix representation of the steady state leads to several exact results for the asymmetric exclusion process. We have discussed here the steady state of the system with open boundary conditions [18], diffusion constant for a system with periodic boundary conditions [20], steady state for two species of particles [19].

There are several other possible generalisations, for example throughout this work we have been concerned with totally asymmetric exclusion although one could equally consider the partially asymmetric exclusion problem where particles can hop either to the right with probability pdt or to the left with probability qdt (with $q = 1 - p$). In that case one can show [18] that replacing (15) by

$$pDE - qED = D + E, \tag{58}$$

still gives the steady state. When $p = \frac{1}{2}$ (the case of symmetric exclusion) it is known that with periodic boundary conditions detailed balance is satisfied, so that qualitatively different behaviour from the asymmetric case might be expected. For $p = \frac{1}{2}$ the diffusion constant has previously been calculated [29] and the dependence on the system size is N^{-1} as opposed to the $N^{-1/2}$ dependence of (44). This is related to the fact that both for the asymmetric and the symmetric cases, the fluctuations of Y_t in the infinite system are subdiffusive ($\langle Y_t^2 \rangle - \langle Y_t \rangle^2 \sim t^{2/3}$ for the asymmetric case and $\sim t^{1/2}$ for the symmetric case).

We have made a numerical calculation of the diffusion constant of a tagged particle on systems of sizes $2 \leq N \leq 10$ for $p = \frac{1}{2}(1+\epsilon)$ (on a ring of N sites with M particles). For ϵ small the first terms of the expansion seem to be given by

$$\begin{aligned}\Delta \;=\; & \frac{(N-M)}{M(N-1)} + \frac{\epsilon^2}{3}\,\frac{(M-1)}{M}\,\frac{(N-M)(N-M-1)}{(N-1)^2} \\ & -\frac{2\epsilon^4}{45}\,\frac{(M-1)(M-2)}{M}\,\frac{(N-M)(N-M-1)(N-M-2)}{(N-1)^2(N-2)} + O(\epsilon^6).\end{aligned} \tag{59}$$

This result is so far only a conjecture based on the analysis of our data. One can see that in the limit of a finite density of particles on an infinite ring ($N \to \infty$ with $M = N\rho$), the terms are of order $1/N, \epsilon^2, \epsilon^4 N \cdots$. Thus it appears that for large N and small ϵ the diffusion constant should be of the form

$$\Delta \sim \frac{1}{N} g(\epsilon^2 N) \tag{60}$$

with $g(x) \sim O(1)$ for $x = 0$ and $g(x) \sim x^{1/2}$ for large x where the function g would describe the crossover between the asymmetric and symmetric processes.

The asymmetric exclusion process is connected to several other problems of interest.

First it can be mapped exactly onto a model of a growing interface in $(1+1)$ dimensions [6] by associating to each configuration $\{\tau_i\}$ of the particles, a configuration of an interface: a particle at a site corresponds to a downwards step of the interface height of one unit whereas a hole corresponds to an upward step of one unit. The heights of the interface are thus defined by

$$h_{i+1} - h_i = 1 - 2\tau_i. \tag{61}$$

The dynamics of the asymmetric exclusion process in which a particle may interchange position with a neighbouring hole to the right, corresponds to an interface dynamics in which a downwards step followed by an upwards step may become an upwards step followed by a downward step. In other words, a growth event occurs at any minimum of the interface height with probability dt, i.e., if $h_i(t) = h_{i+2}(t) = h_{i+1}(t) + 1$ then

$$h_{i+1}(t+dt) = \begin{cases} h_{i+1}(t) & \text{with probability } 1 - dt \\ h_{i+1}(t) + 2 & \text{with probability } dt. \end{cases} \tag{62}$$

Otherwise $h_{i+1}(t)$ remains unchanged. A growth event turns a minimum of the surface height into a maximum thus the system of hopping particles maps onto what is known as a single step growth model meaning that the difference in heights of two neighbouring positions on the the interface is always of magnitude one unit.

Periodic boundary conditions for the particle problem with M particles and $N - M$ holes correspond to an interface satisfying $h_{i+N} = h_i + N - 2M$, i.e., to helical boundary conditions with in average slope $1 - 2M/N$. The case of open boundary conditions corresponds to special growth rules at the boundaries. Because of this equivalence, several results obtained for the asymmetric exclusion process can be translated into exactly computable properties of the growing interface [15].

As is well known [30], the problem of growing interfaces is equivalent to the problem of directed polymers in a random medium (which is known as first-passage percolation in the mathematical literature). It would be of interest to see what kind of quantities could be calculated exactly in the directed polymer problem through the mapping from the asymmetric exclusion process.

As well as the mapping to growth described above, other possible mappings from systems of hopping particles to growth [11] and to other models of physical interest exist. For example, repton models of diffusion of polymer chains and gel electrophoresis may be formulated in terms of exclusion processes with various numbers of species of particles [31, 32, 33, 34]. It would certainly be interesting to see whether these models could be attacked using similar techniques to those outlined here.

Another possible direction in which this work might be extended would be to examine the effects of disorder. Disorder could be introduced in a variety of ways, for example, the hopping rate of each particle could be a quenched random variable. If the hopping rates took only two values and the particles did not overtake each other the disorder would be in the sequence of the particles. For any order of the particles we can describe the steady state in this case as it corresponds to the limit $\delta \to 0$ of the two species model discussed in Section 4. Then the problem would be to analyse the effect of the quenched disorder of the sequence on various properties such as the current and the diffusion constant.

Lastly, a question we feel would be worthwhile answering concerns the relation of the matrix approach to other techniques that are commonly used in statistical mechanics. It is known that in the case of periodic boundaries [7, 8], or of parallel updating [16], the asymmetric exclusion model can be solved by means of the Bethe ansatz. It would certainly be instructive to better understand the link between the traditional Bethe ansatz approach and the matrix formulation we have used here.

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