Introduction to the Standard Model

Lecture 2

Symmetries

$$\begin{tikzcd} & \text{Global symmetry} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \text{Conserved current} \arrow[rr, leftrightarrow] & & \text{Invariance (when acting upon a Lagrangian/Hamiltonian system)} \end{tikzcd}$$

As an example, consider

$$\begin{tikzcd} & \text{homogeneity of spacetime} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \text{momentum conservation} \arrow[rr, leftrightarrow] & & \text{translational invariance} \end{tikzcd}$$

Classification

<u>global:</u>	spacetime symmetries	$\rightarrow$	momentum, angular momentum, spin
	internal symmetries	$\rightarrow$	weak isospin, charge, colour
<u>local:</u>	spacetime symmetries	$\rightarrow$	gravity as a gauge theory
	internal symmetries	$\rightarrow$	gauge theory

Basics of Group theory

(see tutorial for more details)
The Standard Model requires knowledge of the groups, $U(1)$, $SU(2)$, and $SU(3)$, along with some of their matrix representations and associated Lie-algebras.

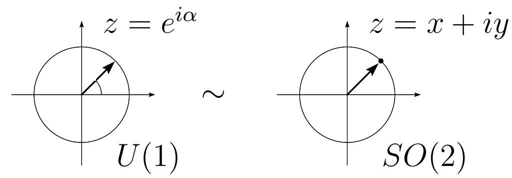
The $U(1)$ group

Each group element of $U(1)$ can be represented by a pure phase factor, $e^{i\alpha}$. The parameter, α, is real and continuous which indicates that $U(1)$ has an infinite set of group elements and is continuous.

Since $e^{i\alpha} = \cos\alpha + i\sin\alpha$, $U(1)$ is isomorphic to 2-by-2 rotation matrices, i.e. elements of $SO(2)$, which also form a Lie-group:

$$e^{i\alpha} = \cos\alpha + i \sin\alpha \quad \underset{\text{isomorphic}}{\sim} \quad \underbrace{\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}}_{\exp\left(i\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)} \in SO(2)$$

To see this
consider the unit circle
in the complex plane:



The $SU(N)$ group

i.) The $SU(N)$ group is defined as the collection of all unitary $N \times N$ matrices U, i.e. $U^{-1} = U^{\dagger}$, with determinant equal to one.

$$U \in SU(N) \;\Rightarrow\; \underbrace{UU^{\dagger} = \mathbb{1}}_{N^2 \text{ relations}},\; \underbrace{\det\big(U\big) = 1}_{1 \text{ relation}}$$

As $U \in \mathbb{C}^{N \times N}$ has N^2 complex entries or $2N^2$ real entries, we are left with $2N^2 - N^2 - 1 = N^2 - 1$ independent parameters.

ii.) Consider the N-vector valued field, $\vec{\psi}$, transforming under $SU(N)$ as

$$\vec{\psi} \to \vec{\psi}' = U\vec{\psi} \qquad \Big(\psi'_j = U_{jl}\psi_l \;\text{ with } j, l = 1, \ldots, N\Big)$$

$\vec{\psi}$ is in the fundamental representation. We also note that $\vec{\psi}^{\dagger}\vec{\psi}$ is invariant under an $SU(N)$ transformation since

$$\begin{aligned} \vec{\psi}\vec{\psi} \to \vec{\psi}'^{\dagger}\vec{\psi}' &= \big(U\vec{\psi}\big)^{\dagger} U\vec{\psi} \\ &= \vec{\psi}^{\dagger} U^{\dagger} U \vec{\psi} \\ &= \vec{\psi}^{\dagger}\vec{\psi} \end{aligned}$$

iii.) A group element U can be expressed as an exponential

$$U(\Lambda_1, \ldots, \Lambda_{N^2-1}) = \exp\left(i \sum_{a=1}^{N^2-1} \Lambda_a \mathrm{T}_a\right) = \lim_{n\to\infty}\left(\mathbb{1} + i\frac{\Lambda_a}{n}\mathrm{T}_a\right)^n$$

where Λ_a are real-valued and continuous, and $\mathrm{T}_{a=1,\ldots,N^2-1}$ are called the *generators* of the group. The conditions imposed by i.) and ii.) above restricts the T's:

a.)

$$\begin{aligned}U^\dagger U = \mathbb{1} \Rightarrow & \left(e^{i\varepsilon_a \mathrm{T}_a}\right)^\dagger \left(e^{i\varepsilon_b \mathrm{T}_b}\right)\\
&= \left(\mathbb{1} + i\varepsilon_a \mathrm{T}_a + \mathcal{O}(|\varepsilon_a|^2)\right)^\dagger \left(\mathbb{1} + i\varepsilon_b \mathrm{T}_b + \mathcal{O}(|\varepsilon_b|^2)\right)\\
&= \left(\mathbb{1} - i\varepsilon_a \mathrm{T}_a^\dagger + \cdots\right)\left(\mathbb{1} + i\varepsilon_b \mathrm{T}_b + \cdots\right)\\
&= \mathbb{1} + i\varepsilon_b\left(\mathrm{T}_b - \mathrm{T}_b^\dagger\right) + \cdots\end{aligned}$$

We see that the generators must be Hermitian: $\mathrm{T}_b = \mathrm{T}_b^\dagger$.

b.)

$$\begin{aligned}\det U = 1 \Rightarrow \ & \det\left(e^{i\varepsilon_a \mathrm{T}_a}\right)\\
&= 1 + i\varepsilon_a \mathrm{tr}\left(\mathrm{T}_a\right) + \cdots\end{aligned}$$

This implies that the generators must be traceless: $\mathrm{tr}\left(\mathrm{T}_a\right) = 0$.

Lie Algebra

The generators of $SU(N)$ obey an important property. Evaluate

$$U = U_2^{-1} U_1^{-1} U_2 U_1 \in SU(N) \tag{1}$$

by using

$$\begin{aligned}U &= \mathbb{1} + i\lambda_c \mathrm{T}_c + \cdots\\
U_1 &= \mathbb{1} + i\varepsilon_a \mathrm{T}_a - \frac{1}{2}(\varepsilon \cdot T)^2 + \cdots \Leftrightarrow U_1^{-1} = U_1^\dagger = \mathbb{1} - i(\varepsilon \cdot T) - \frac{1}{2}(\varepsilon \cdot T)^2 + \cdots\\
U_2 &= \mathbb{1} + i\delta_b \mathrm{T}_b - \frac{1}{2}(\delta \cdot T)^2 + \cdots \Leftrightarrow U_2^{-1} = U_2^\dagger = \mathbb{1} - i(\delta \cdot T) - \frac{1}{2}(\delta \cdot T)^2 + \cdots\end{aligned}$$

One gets

$$RHS \text{ of } (1) = \mathbb{1} - i(\delta \cdot T + \varepsilon \cdot T - \delta \cdot T - \varepsilon \cdot T) + \varepsilon_a \delta_b\left(\mathrm{T}_a \mathrm{T}_b - \mathrm{T}_b \mathrm{T}_a\right) + \cdots$$

compared to the LHS of (1)

$$\mathbb{1} + i\lambda_c \mathrm{T}_c + \cdots = \mathbb{1} + \varepsilon_a \delta_b\left[\mathrm{T}_a, \mathrm{T}_b\right] + \cdots$$

results into

$$\left[\mathrm{T}_a, \mathrm{T}_b\right] = i f_{abc} \mathrm{T}_c \qquad f_{abc} \in \mathbb{R}$$

The f_{abc}'s are anti-symmetric *structure constants* of the Lie group. The *generators*, T_a, with such a property, form the so-called *Lie-algebra* associated to the Lie-group.

$$U = \underbrace{\exp\left(i\Lambda_a \mathrm{T}_a\right)}_{\text{whole group}} = \underbrace{\mathbb{1} + i\Lambda_a \mathrm{T}_a + \cdots}_{\substack{\text{local elements define}\\ \text{the properties}\\ \text{of the whole group}}}$$

Note The Lie group forms a compact manifold; the algebra is defined on the tangent to the unit element.

We choose a normalisation, $\mathrm{tr}\big(\mathrm{T}_a \mathrm{T}_b\big) = T_R \delta_{ab}$ with $T_R = \frac{1}{2}$(convention) such that

$$f_{abc} = -2i\mathrm{tr}\Big(\big[\mathrm{T}_a, \mathrm{T}_b\big]\mathrm{T}_c\Big)$$

Important Examples

$SU(2)$

The generators are proportional to the Pauli matrices: $\mathrm{T}_a = \frac{1}{2}\sigma^a$.

The algebra is given as: $\left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2}\right] = i\varepsilon^{abc}\frac{\sigma^c}{2}$ where the structure constants are the defined by the totally anti-symmetric epsilon tensor ε^{abc}. Remember that

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$SU(3)$

The algebra is: $\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right] = if^{abc}\frac{\lambda^c}{2}$ where the λ^a's are the Gell-Mann matrices,

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The structure constants are:

$f^{123} = 1$

$f^{147} = f^{246} = f^{345} = f^{316} = f^{257} = f^{637} = \frac{1}{2}$

$f^{458} = f^{678} = \frac{\sqrt{3}}{2}$

$f^{\mathrm{other}} = 0$

Note that we can see the $SU(2)$ subgroup in $SU(3)$:

$$\lambda^1 = \begin{pmatrix} \sigma^1 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} \sigma^2 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} \sigma^3 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Exercise check all the above