Introduction to the Standard Model

Lecture 5: Gauge Theories

First some history[1]

- Weyl (1918) proposed space-time dependent length scales
  $\rightarrow$ scaling transformations $\sim e^{\Lambda(x)}$
- Fock, Weyl (1927-1929) in QED, global phases of wave functions non-observable, 'minimal coupling' $\hat{p}_\mu \rightarrow \hat{p}_\mu + A_\mu Q$ where A_μ is gauge field
  $\rightarrow$ amounts to 'local' phases $\sim e^{i\Lambda(x)}$.
- Yang, Mills (1954) applied this to non-Abelian groups, *e.g.* $SU(2)$,
  $\rightarrow$ leads to 'matrix-valued phases' $e^{i\Lambda_a(x)\sigma^a/2}$.

As we shall see the *local* realisation of symmetry transformations induces dynamics, i.e. interaction terms.

Field Theories with an abelian gauge group

i) Global $U(1)$ symmetry: Consider a free charged scalar/fermion field, given by the respective Lagrangians:

$$\begin{aligned}\mathcal{L}^0_{\text{KG}} &= \partial_\mu\phi(x)^*\partial^\mu\phi(x) - m^2\phi(x)^*\phi(x)\\ \mathcal{L}^0_{\text{D}} &= i\overline{\psi}(x)\partial\!\!\!/\psi(x) + m\overline{\psi}(x)\psi(x)\end{aligned}$$

Both Lagrangians are invariant under a global $U(1)$ symmetry:

$$\begin{aligned}\phi \rightarrow \phi' &= U(\Lambda)\phi\\ \psi \rightarrow \psi' &= U(\Lambda)\psi\end{aligned}$$

where $U(\Lambda) = e^{i\Lambda Q}$ and $\Lambda \in \mathbb{R}$ is a constant, i.e. a *global* parameter.
Note: we have assumed, for simplicity, that ϕ and ψ both have the same charge Q.
We can derive the Noether Currents respectively as

$$\begin{aligned}\widehat{J}^\mu_{\text{KG}} &= iQ(\phi(x)^*\partial^\mu\phi(x) - \phi(x)\partial^\mu\phi(x)^*)\\ \widehat{J}^\mu_{\text{D}} &= Q\overline{\psi}(x)\gamma^\mu\psi(x)\end{aligned}$$

Note: The Noether currents and the respective charges are conserved due to the global symmetry.

[1] An excellent recent article can be found on the internet: L. O'Raifeartaigh and N. Straumann, "Gauge theory: Historical origins and some modern developments," Rev. Mod. Phys. **72** (2000) 1.

ii) Local $U(1)$ symmetry: Let $\Lambda \to \Lambda(x^\mu)$ such that $\Lambda(x^\mu)$ is now a function of spacetime. In the case of the complex scalar field

$$\begin{aligned} \phi &\to \phi' = U(\Lambda(x))\phi \\ \phi^* &\to \phi^{*\prime} = \phi^* U^\dagger(\Lambda(x)) \end{aligned}$$

However, this does not allow $\mathcal{L}^0_{\text{KG}}$ to be invariant under local $U(1)$ symmetry since

$$\begin{aligned} \partial^\mu \phi(x) \to \partial^\mu \phi' &= \partial^\mu \big(U(\Lambda(x))\phi\big) \\ &= \big(\partial^\mu U(\Lambda(x))\big)\phi + U(\Lambda(x))\partial^\mu \phi \\ &\neq U(\Lambda(x))\partial^\mu \phi \quad \text{which is needed for invariance} \end{aligned}$$

A generalisation of the differential operator is needed, a *covariant* derivative. Specifically, we require

$$\begin{aligned} &D^\mu \phi \to D^{\mu\prime}\phi' \stackrel{!}{=} U D^\mu \phi \\ &(D^\mu \phi)^* \to (D^{\mu\prime}\phi')^* \stackrel{!}{=} D^\mu \phi U^\dagger \\ &(D^{\mu\prime}\phi')^*(D^{\mu\prime}\phi') \stackrel{!}{=} (D^\mu\phi)^* U^\dagger U (D^\mu \phi) = (D^\mu\phi)^*(D^\mu\phi) \end{aligned}$$

We choose

$$\partial^\mu \to D^\mu \equiv \partial^\mu + iQA^\mu$$

as our ansatz.

$$D^{\mu\prime}\phi' = (\partial^\mu + iQA^{\mu\prime})\phi' \stackrel{!}{=} UD^\mu\phi = U(\partial^\mu + iQA^\mu)\phi$$

which can be rewritten as

$$[(\partial^\mu U)U^{-1} + iQA^{\mu\prime}]U\phi = UiQA^\mu(U^{-1})U\phi$$

This implies that we need a 4-vector that transforms as

$$A^{\mu\prime} = UA^\mu U^{-1} + i/Q(\partial^\mu U)U^{-1}$$

In the case of $U(1)$, U is just a phase: $U = e^{iQ\Lambda(x)}$ hence $UA^\mu = UA^\mu$ (trivial) and $\partial^\mu U = iQ(\partial^\mu \Lambda)U$. Then we find that

$$A^{\mu\prime} = A^\mu - \partial^\mu \Lambda$$

which is just a gauge transformation of a vector potential A^μ.
In other words, if one couples a globally $U(1)$ symmetric theory to a gauge field A^μ by the covariant derivative $\partial^\mu \to D^\mu \equiv \partial^\mu + iQA^\mu$, one ends up with a *locally* symmetric $U(1)$ theory. This way of coupling a gauge potential to a matter field is traditionally called *principle of minimal coupling*. In the context of gauge theories it is deduced from the local invariance property.

The free Klein-Gordon Lagrangian has to be modified in this respect

$$\begin{aligned} \mathcal{L}^0_{\text{KG}} \to \mathcal{L}_{\text{KG}} &= (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi \\ &= [(\partial_\mu - iQA_\mu)\phi^*](\partial^\mu + iQA^\mu)\phi - m^2\phi^*\phi \\ &= \partial_\mu\phi^*\partial^\mu\phi + iQ(\partial_\mu\phi^*)\phi A^\mu - iQA_\mu\phi^*\partial^\mu\phi + Q^2A_\mu A^\mu\phi^*\phi - m^2\phi^*\phi \\ &= \mathcal{L}^0_{\text{KG}} + \mathcal{L}_{\text{Interaction}} \end{aligned}$$

where we define $\mathcal{L}_{\text{Interaction}}$ as

$$\mathcal{L}_{\text{Interaction}} \equiv \widehat{J}^{\mu}_{\text{KG}} A_\mu + Q^2 A_\mu A^\mu \phi^* \phi$$

Now we see that our theory which is invariant under local gauge transformations is promoted to an interacting theory. Hence we conclude

$$\boxed{\text{Local gauge symmetry} \Rightarrow \text{matter-gauge field interaction}}$$

ie locally symmetric theories induce uniquely defined interaction properties.

The dynamics of the gauge field A^μ is induced in the standard way:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = -(i/Q)[D^\mu, D^\nu] \quad \text{(tutorial)}$$

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

where $F^{\mu\nu}$ is constructed to be gauge invariant as it is symmetric under a $U(1)$ gauge transformation: $A^\mu \to A^{\mu\prime} = A^\mu - \partial^\mu \Lambda$.
The full Lagrangian to describe a free complex scalar field is thus given by

$$\mathcal{L} = \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{KG}} = \mathcal{L}_{\text{Maxwell}} + \mathcal{L}^0_{\text{KG}} + \mathcal{L}_{\text{Interaction}}$$

This is known as *Scalar Electrodynamics*. It is the first example of a non-trivial field theory based on a commuting symmetry group and is hence an example of an Abelian Gauge Theory.

The Field Equations

The field equations are obtained by the principle of least action $\delta S = 0$ from the action $S = \int d^4x\, \mathcal{L}$ and for the gauge field A_μ turn out to be:

$$\partial_\mu F^{\mu\nu} = \partial^2 A^\nu - \partial^\nu(\partial_\mu A^\mu) = J^\nu \quad ,$$

which along with the Bianchi Identity

$$\partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} + \partial^\rho F^{\mu\nu} = 0$$

lead to the Maxwell equations. The gauge invariant field equations for the complex scalar field are

$$\begin{aligned}(D_\mu D^\mu + m^2)\phi &= 0 \\ (D_\mu D^\mu + m^2)\phi^* &= 0\end{aligned}$$

with

$$\begin{aligned} J^\nu &= -\frac{\partial \mathcal{L}_{\text{Int}}}{\partial A_\nu} \\ &= iQ(\phi^* D^\nu \phi - \phi D^\nu \phi^*) \\ &= \widehat{J}_{\text{KG}} - 2Q^2 \phi^* \phi A^\mu \end{aligned}$$

We can also see now that J^μ is the covariant version of the Noether Current:

$$\widehat{J}_{\text{KG}} \xrightarrow{\partial^\mu \to D^\mu} J^\nu$$