Electromagnetism - Lecture 13

Waves in Insulators

- Refractive Index & Wave Impedance
- Dispersion
- Absorption
- Models of Dispersion & Absorption
- The Ionosphere
- Example of Water

Maxwell's Equations in Insulators

Maxwell's equations are modified by ϵ_r and μ_r

- Either put ϵ_r in front of ϵ_0 and μ_r in front of μ_0
- Or remember $\mathbf{D} = \epsilon_r \epsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_r \mu_0 \mathbf{H}$

Solutions are wave equations:

$$\nabla^2 \mathbf{E} = \epsilon_r \epsilon_0 \mu_r \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{\epsilon_r \mu_r}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

The effect of ϵ_r and μ_r is to **change the wave velocity**:

$$v = \frac{1}{\sqrt{\epsilon_r \epsilon_0 \mu_r \mu_0}} = \frac{c}{\sqrt{\epsilon_r \mu_r}}$$

Refractive Index & Wave Impedance

For non-magnetic materials with $\mu_r = 1$:

$$v = \frac{c}{\sqrt{\epsilon_r}} = \frac{c}{n} \qquad n = \sqrt{\epsilon_r}$$

The **refractive index** n is usually slightly larger than 1

Electromagnetic waves travel slower in dielectrics

The **wave impedance** is the ratio of the field amplitudes:

$$Z = E/H \qquad \text{in units of } \Omega = V/A$$

In vacuo the impedance is a constant: $\qquad Z_0 = \mu_0 c = 377\Omega$

In an insulator the impedance is:

$$Z = \mu_r \mu_0 v = \frac{\mu_r \mu_0 c}{\sqrt{\epsilon_r \mu_r}} = \sqrt{\frac{\mu_r}{\epsilon_r}} Z_0$$

For non-magnetic materials with $\mu_r = 1$: $\qquad Z = Z_0/n$

Notes:

Diagrams:

Energy Propagation in Insulators

The Poynting vector $\mathbf{N} = \mathbf{E} \times \mathbf{H}$ measures the energy flux

Energy flux is energy flow per unit time through surface normal to direction of propagation of wave:

$$\frac{\partial U}{\partial t} = \int_A \mathbf{N}.\mathbf{dS} \qquad \text{Units of } \mathbf{N} \text{ are Wm}^{-2}$$

In vacuo the amplitude of the Poynting vector is:

$$N_0 = \frac{1}{2} E_0^2 \sqrt{\frac{\epsilon_0}{\mu_0}} = \frac{1}{2} \frac{E_0^2}{Z_0}$$

In an insulator this becomes:

$$N = N_0 \sqrt{\frac{\epsilon_r}{\mu_r}} = \frac{1}{2} \frac{E_0^2}{Z}$$

The energy flux is proportional to the square of the amplitude, and inversely proportional to the wave impedance

Dispersion

Dispersion occurs because the dielectric constant ϵ_r and refractive index n are functions of frequency ω

Waves with different frequencies propagate with different velocities

For a particular frequency the **phase velocity** is:

$$v_p(\omega) = \frac{\omega}{k} = \frac{c}{n(\omega)}$$

For a *wavepacket* containing a small range of frequencies $\Delta\omega << \omega$ the **group velocity** is:

$$v_g(\omega) = \frac{d\omega}{dk} = \frac{c}{(n + \omega dn/d\omega)}$$

Energy transmission in a wavepacket is described by group velocity!

For most insulators $dn/d\omega > 0$, $n > 1$ and $v_g < c$

Absorption

Absorption can be represented by a complex dielectric constant:

$$\epsilon_r = \epsilon_1 - i\epsilon_2$$

The refractive index is also complex:

$$n = \sqrt{(\epsilon_1 - i\epsilon_2)} = n_1 - in_2 \qquad n_1 = \sqrt{\epsilon_1} \quad n_2 = \frac{\epsilon_2}{2\sqrt{\epsilon_1}}$$

where we assume that $\epsilon_2 \ll \epsilon_1$

Plane wave solutions have a complex wavenumber $k = k_1 - ik_2$:

$$E = E_0 e^{i(\omega t - kz)} \qquad E = E_0 e^{-k_2 z} e^{i(\omega t - k_1 z)}$$

The imaginary part of the wavenumber gives an exponential attenuation coefficient in the amplitude

Phase Velocity and Attenutation

The phase velocity of the wave is given by the real parts of the dielectric constant or refractive index:

$$v = \frac{\omega}{k_1} = \frac{c}{n_1} = \frac{c}{\sqrt{\epsilon_1}}$$

The attenuation length of the wave is inversely proportional to the imaginary part of the refractive index:

$$\alpha = \frac{1}{k_2} = \frac{c}{\omega n_2} = \frac{2c\sqrt{\epsilon_1}}{\omega \epsilon_2}$$

where we assume the absorption is small $\epsilon_2 \ll \epsilon_1$

Harmonic Oscillator Model

Equation of motion of an electron in an external electric field:

$$m_e \left(\frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x \right) = -e\mathbf{E}$$

where γ is a damping term due to other forces on the electron and ω_0 is the natural *resonant frequency* of the electron

An oscillating electric field causes simple harmonic motion:

$$\mathbf{E} = E_0 e^{i(kz-\omega t)} \qquad x = x_0 e^{i\omega t}$$

$$x_0 = \frac{-eE_0}{m_e[(\omega_0^2 - \omega^2) + i\omega\gamma]}$$

An oscillating electron can be described by an oscillating electric dipole moment:

$$\mathbf{p} = -e\mathbf{x} = -ex_0 e^{i\omega t}\hat{\mathbf{x}}$$

Model of Dispersion & Absorption

The oscillating electrons create a polarization $\mathbf{P} = N_e \mathbf{p}$

The electric susceptibility χ_E is:

$$\chi_E = \frac{\mathbf{P}}{\epsilon_0 \mathbf{E}} = \frac{N_e e^2}{m_e \epsilon_0 [(\omega_0^2 - \omega^2) + i\omega\gamma]}$$

and the dielectric constant is:

$$\epsilon_r(\omega) = 1 + \frac{N_e e^2}{m_e \epsilon_0 [(\omega_0^2 - \omega^2) + i\omega\gamma]}$$

The real and imaginary parts are:

$$\epsilon_1 = 1 + \frac{N_e e^2 (\omega_0^2 - \omega^2)}{m_e \epsilon_0 [(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2]} \qquad \epsilon_2 = \frac{N_e e^2 \omega\gamma}{m_e \epsilon_0 [(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2]}$$

The Ionosphere

The ionosphere is a region of the upper atmosphere that contains a **plasma** of free electrons

It can be described by the harmonic oscillator model if we assume $\omega \gg \omega_0$ and neglect damping $\gamma = 0$:

$$\epsilon_r(\omega) = 1 - \frac{\omega_P^2}{\omega^2}$$

where the *plasma frequency* ω_P depends on the electron density:

$$\omega_P = \sqrt{\frac{N_e e^2}{m_e \epsilon_0}}$$

There is a **dispersion relation** between k and ω:

$$k = \frac{\sqrt{\omega^2 - \omega_P^2}}{c}$$

Reflection of Waves by the Ionosphere

For frequencies $\omega < \omega_P$, $\epsilon_r < 0$ and k is purely imaginary

Waves with $\omega < \omega_P$ do not propagate through the ionosphere

The plasma is effectively a conductor and totally reflects the waves

For frequencies $\omega > \omega_P$, $\epsilon_r > 0$ and k is real

Waves with $\omega > \omega_P$ propagate with no attenuation

The plasma is effectively an insulator with phase velocity $v_p > c$ and group velocity $v_g < c$:

$$v_p = c\sqrt{\frac{1}{(1 - \omega_P^2/\omega^2)}} \qquad v_g = \frac{c^2}{v_p}$$

Absorption in Molecular Materials

In molecular materials there can be many different **resonant frequencies** ω_0 associated with rotational and vibrational states

At the resonances there is large absorption

The width of a resonance is controlled by the damping term γ

The Q-factor is:

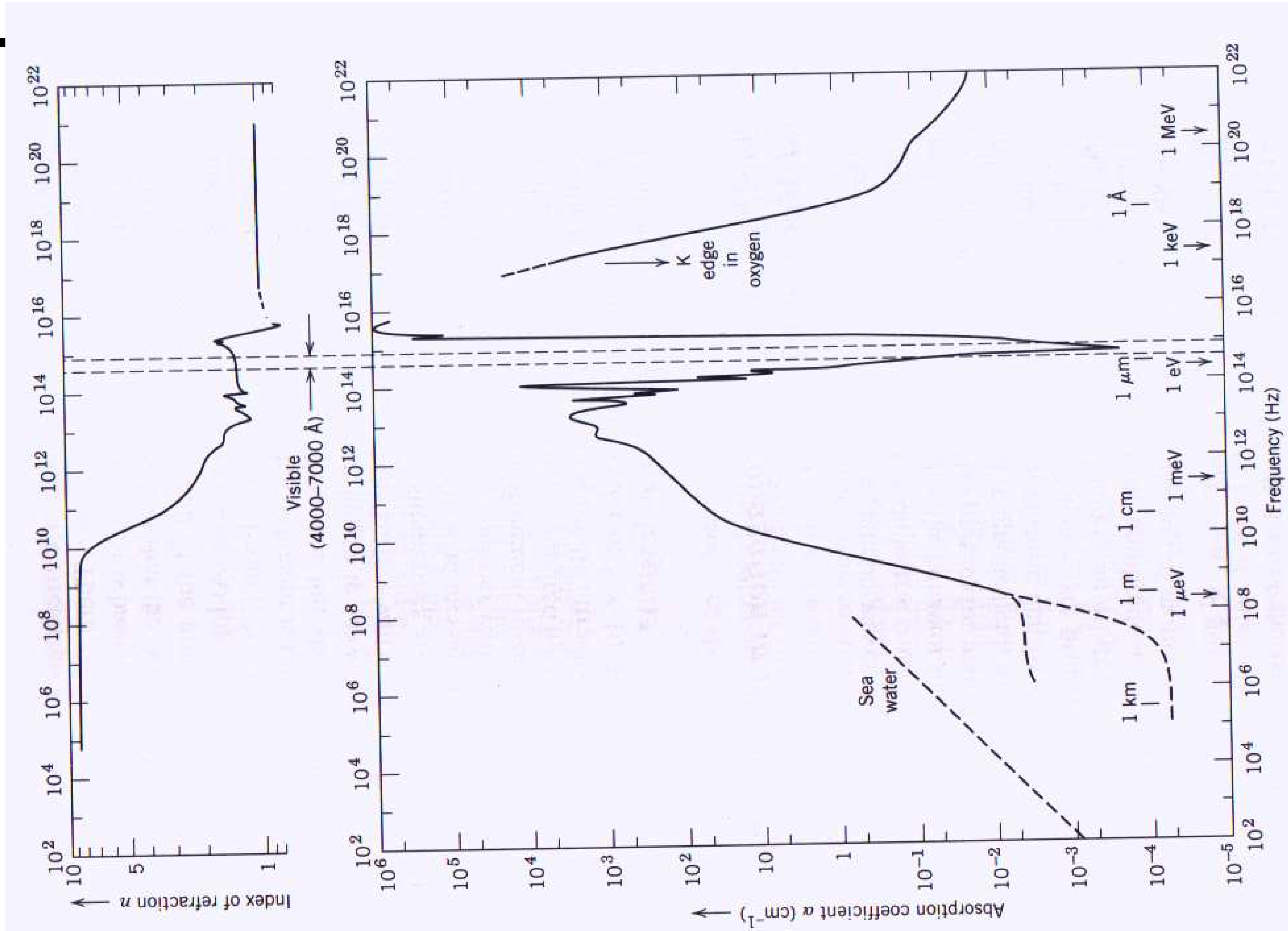
$$Q = \frac{\omega_0}{2|\Delta\omega_{1/2}|} = \frac{\omega_0}{\gamma}$$

Example of Polar Dielectric (Water)

- Low frequencies $\omega \ll 10^{10}$Hz:
  No resonances. Negligible absorption. Static limit $\epsilon_1 \rightarrow 81$.
  These conclusions are modified by the presence of conducting ions in salt water.

- Microwaves $\omega \approx 10^{11}$Hz:
  Rotational states lead to large absorption bands.
  Thermal motion disupts alignment of molcular dipole moments.
  ϵ_1 decreases as a function of ω.

- Infrared $\omega = 10^{13} - 10^{14}$Hz:
  Vibrational states lead to large absorption bands.
  These have narrower widths than rotational states.
  ϵ_1 and ϵ_2 vary rapidly.



Jackson (Figure 7.9, P.315) - refractive index $n = \sqrt{\epsilon_1}$ (left) and absorption coefficient α (right) of water as function of ω

- Visible light $\omega = 4 - 8 \times 10^{14}$Hz:
  Transparent due to large hole in absorption coefficient.

- Ultraviolet $\omega = 10^{15} - 10^{16}$Hz
  Absorption is large due to collective excitations of electrons known as *plasmons*.
  Can be modelled by a plasma frequency ω_P.

- High frequencies $\omega \gg \omega_P$:
  $\epsilon_1 \approx 1$ and absorption is negligible.