Electromagnetism - Lecture 18

Relativity & Electromagnetism

- Special Relativity
- Current & Potential Four Vectors
- Lorentz Transformations of **E** and **B**
- Electromagnetic Field Tensor
- Lorentz Invariance of Maxwell's Equations

Special Relativity in One Slide

Space-time is a **four-vector**: $x^\mu = [ct, \mathbf{x}]$

Four-vectors have Lorentz transformations between two frames with uniform relative velocity v:

$$x' = \gamma(x - \beta ct) \qquad ct' = \gamma(ct - \beta x)$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$
The factor γ leads to time dilation and length contraction

Products of four-vectors are Lorentz invariants:

$$x^\mu x_\mu = c^2t^2 - |\mathbf{x}|^2 = c^2t'^2 - |\mathbf{x}'|^2 = s^2$$

The maximum possible speed is c where $\beta \to 1$, $\gamma \to \infty$.

Electromagnetism predicts waves that travel at c in a vacuum!
The laws of Electromagnetism should be Lorentz invariant

Charge and Current

Under a Lorentz transformation a static charge q at rest becomes a charge moving with velocity $\mathbf{v}$. This is a current!

A static charge density ρ becomes a current density $\mathbf{J}$

N.B. Charge is conserved by a Lorentz transformation

The **charge/current four-vector** is:

$$J^\mu = \rho \frac{dx^\mu}{dt} = [c\rho, \mathbf{J}]$$

The full Lorentz transformation is:

$$J'_x = \gamma(J_x - v\rho) \qquad \rho' = \gamma(\rho - \frac{v}{c^2} J_x)$$

Note that the γ factor can be understood as a length contraction or time dilation affecting the charge and current densities

Notes:

Diagrams:

Electrostatic & Vector Potentials

A reminder that:

$$V = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho}{r}d\tau \qquad \mathbf{A} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}}{r}d\tau$$

A static charge density ρ is a source of an electrostatic potential V

A current density $\mathbf{J}$ is a source of a magnetic vector potential $\mathbf{A}$

Under a Lorentz transformation a V becomes an $\mathbf{A}$:

$$A'_x = \gamma(A_x - \frac{v}{c^2}V) \qquad V' = \gamma(V - vA_x)$$

The **potential four-vector** is:

$$A^\mu = \left[\frac{V}{c}, \mathbf{A}\right]$$

Relativistic Versions of Equations

Continuity equation:

$$\nabla.\mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \qquad \partial_\mu J^\mu = 0$$

This shows that charge conservation is Lorentz invariant!

Lorentz gauge condition:

$$\frac{1}{c}\frac{\partial V}{\partial t} + \nabla.\mathbf{A} = 0 \qquad \partial_\mu A^\mu = 0$$

Poisson's equations:

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \qquad \nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$\partial_\mu^2 A^\mu = -\mu_0 J^\mu$$

Electric and Magnetic Fields

The Lorentz force on a moving charge is:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

A static point charge is a source of an $\mathbf{E}$ field

A moving charge is a current source of a $\mathbf{B}$ field

Whether a field is $\mathbf{E}$ *or* $\mathbf{B}$ *depends on the observer's frame*

Going from the rest frame to a frame with velocity $\mathbf{v}$:

$$\mathbf{B}' = \frac{1}{c^2}\mathbf{v} \times \mathbf{E}$$

Going from a moving frame to the rest frame:

$$\mathbf{E}' = -\mathbf{v} \times \mathbf{B}$$

This formula was already derived from Induction (Lecture 6)

Notes:

Diagrams:

Lorentz transformations of E and B

The fields in terms of the potentials are:

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla V \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz transformation of potentials:

$$V' = \gamma(V - vA_x) \qquad A'_x = \gamma(A_x - \frac{v}{c^2}V)$$

Using this transformation and the Lorentz gauge condition the transformations of the electric and magnetic fields are:

$$E'_x = E_x \qquad E'_y = \gamma(E_y - vB_z) \qquad E'_z = \gamma(E_z + vB_y)$$

$$B'_x = B_x \qquad B'_y = \gamma(B_y + \frac{v}{c^2}E_z) \qquad B'_z = \gamma(B_z - \frac{v}{c^2}E_y)$$

Fields of Highly Relativistic Charge

A charge at rest has $\mathbf{B} = 0$ and a spherically symmetric $\mathbf{E}$ field

A highly relativistic charge has $\beta \to 1$, $\gamma \gg 1$

The electric field is in the $\hat{\mathbf{r}}$ direction transverse to $\mathbf{v}$:

$$E'_x = E_x \ll |\mathbf{E}'| \qquad E'_y = \gamma E_y \qquad E'_z = \gamma E_z$$

The magnetic field is in the $\hat{\phi}$ direction transverse to $\mathbf{v}$:

$$B'_x = 0 \qquad B'_y = \gamma \frac{v}{c^2} E_z \qquad B'_z = -\gamma \frac{v}{c^2} E_y$$

The electric and magnetic fields are perpendicular to each other with an amplitude ratio $|\mathbf{B}'| = |\mathbf{E}'|/c$

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Electromagnetic Field Tensor

The electric and magnetic fields can be expressed as components of an electromagnetic field tensor:

$$F^{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$$

where $A = [V/c, \mathbf{A}]$ and $x = [ct, \mathbf{x}]$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

Maxwell's Equations in terms of $F^{\mu\nu}$

Source equations:

$$\frac{\partial F^{\mu\nu}}{\partial x_\nu} = J^\mu$$

M1:	$\nabla.\mathbf{E} = \rho/\epsilon_0$	$\mu = 0,\ \nu = (1,2,3)$
M4:	$\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \epsilon_0 \partial \mathbf{E}/\partial t)$	$\mu = 1,\ \nu = (2,3,0)$
		(similarly for $\mu = 2,3$)

No-source equations:

$$\frac{\partial F^{\mu\nu}}{\partial x_\sigma} + \frac{\partial F^{\sigma\mu}}{\partial x_\nu} + \frac{\partial F^{\nu\sigma}}{\partial x_\mu} = 0$$

M2:	$\nabla.\mathbf{B} = 0$	$(\mu,\nu,\sigma) = (1,2,3)$
M3:	$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$	$(\mu,\nu,\sigma) = (0,1,2)(3,0,1)(2,3,0)$

Lorentz Invariance of Maxwell's Equations

As an example we consider the Lorentz transformation of M2:

$$\nabla.\mathbf{B} = 0 \qquad \Rightarrow \qquad \nabla'.\mathbf{B}' = 0$$

$$\frac{\partial B_x}{\partial x'} + \gamma\left(\frac{\partial B_y}{\partial y'} - \frac{\beta}{c}\frac{\partial E_z}{\partial y'} + \frac{\partial B_z}{\partial z'} + \frac{\beta}{c}\frac{\partial E_y}{\partial z'}\right) = 0$$

We note that:

$$x = \gamma(x' + \beta ct') \qquad ct = \gamma(ct' + \beta x') \qquad \frac{\partial B_x}{\partial x'} = \gamma\left(\frac{\partial B_x}{\partial x} + \frac{\beta}{c}\frac{\partial B_x}{\partial t}\right)$$

Substituting into the previous equation and simplifying:

$$\gamma\nabla.\mathbf{B} + \frac{\gamma\beta}{c}\left(\frac{\partial B_x}{\partial t} - \frac{\partial E_z}{\partial y} + \frac{\partial E_y}{\partial z}\right) = 0$$

The second bracket is zero from M3: $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$

Hence $\nabla.\mathbf{B} = \mathbf{0}$ and $\nabla'.\mathbf{B}' = \mathbf{0}$ as required for Lorentz invariance.

Notes:

Diagrams: