Electromagnetism - Lecture 8

Maxwell's Equations

- Continuity Equation
- Displacement Current
- Modification to Ampère's Law
- Maxwell's Equations in Vacuo
- Solution of Maxwell's Equations
- Introduction to Electromagnetic Waves

Continuity Equation

Charge conservation is a fundamental law of physics

Moving a charge from r_1 to r_2:
- decreases charge density $\rho(r_1)$ and increases $\rho(r_2)$
- requires a current I between r_1 and r_2

This conservation law is written as a continuity equation:

$$\mathrm{I} = \oint_A \mathbf{J}.\mathbf{dS} = -\frac{\partial}{\partial t}\int_V \rho d\tau$$

Using the divergence theorem we obtain the differential form:

$$\nabla.\mathbf{J} = -\frac{\partial \rho}{\partial t}$$

At any point in space the divergence of the current density is proportional to the time-derivative of the charge density

Displacement Current

Starting from the differential form of Ampère's law:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

We can take the divergence of both sides:

$$\nabla . \nabla \times \mathbf{B} = \mu_0 \nabla . \mathbf{J} = 0$$

since "div curl = 0"

$\nabla . \mathbf{J} = 0$ *is inconsistent with the continuity equation if there are time-varying charge densities*

$\Rightarrow$ Ampère's law is only correct in the *electrostatic* limit

Solved by adding a **displacement current** to the RHS:

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J} + \mathbf{J_D}) \qquad \nabla . \mathbf{J_D} = \frac{\partial \rho}{\partial t}$$

Modified Version of Ampère's law

We can use Gauss's Law to replace ρ with $\nabla.\mathbf{E}$ in the expression for the displacement current:

$$\nabla.\mathbf{J_D} = \epsilon_0 \frac{\partial(\nabla.\mathbf{E})}{\partial t}$$

Removing the divergences:

$$\mathbf{J_D} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

The displacement current is the time derivative of the electric field

Modified version of Ampère's law with displacement current:

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t})$$

Maxwell's Equations

The laws of electromagnetism are summarized in four differential equations (M1-4) known as Maxwell's equations:

Gauss's Law for **E**:	$\nabla.\mathbf{E} = \rho/\epsilon_0$	M1
Gauss's Law for **B**:	$\nabla.\mathbf{B} = 0$	M2
Faraday's Law of Induction:	$\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$	M3
Modified Ampère's Law:	$\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \epsilon_0\partial\mathbf{E}/\partial t)$	M4

"Maxwell's equations are used constantly and universally in the solution of a variety of practical problems"
(Halliday & Resnick P.566)

Solution of Maxwell's Equations in Vacuo

In a vacuum no charges and currents: $\rho = 0$ and $|\mathbf{J}| = 0$

Take the curl of Maxwell's Equation M3

Then substitute for $\nabla \times \mathbf{B}$ using M4:

$$\nabla \times \nabla \times \mathbf{E} = -\frac{\partial(\nabla \times \mathbf{B})}{\partial t}$$

$$\nabla \times \nabla \times \mathbf{E} = -\epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Alternatively take curl of Maxwell's equation M4

Then substitute for $\nabla \times \mathbf{E}$ using M3:

$$\nabla \times \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial(\nabla \times \mathbf{E})}{\partial t}$$

$$\nabla \times \nabla \times \mathbf{B} = -\epsilon_0 \mu_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

Notes:

Diagrams:

Use the vector calculus identity "curlcurl = grad div - delsquared":

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla.\mathbf{E}) - \nabla^{\mathbf{2}}\mathbf{E}$$

In the absence of free charges $\rho = 0$, M1 gives $\nabla.\mathbf{E} = 0$:

$$\nabla \times \nabla \times \mathbf{E} = -\nabla^{\mathbf{2}}\mathbf{E} = -\epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2}$$

Solutions of Maxwell's Equations in vacuo are **wave equations**:

$$\nabla^{\mathbf{2}}\mathbf{E} = \epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2}$$

$$\nabla^{\mathbf{2}}\mathbf{B} = \epsilon_0\mu_0\frac{\partial^2\mathbf{B}}{\partial t^2}$$

Velocity of waves is:

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 3 \times 10^8 m/s$$

Description of Electromagnetic Waves

The solutions of Maxwell's Equations in vacuo are **electromagnetic (EM) waves**

- EM waves can travel through a vacuum
- In vacuo all EM waves travel at the speed of light c
- EM waves can have any frequency $\nu = 0$ to ∞
- EM waves have oscillating $\mathbf{E}$ and $\mathbf{B}$ fields
- There are *two* different *polarization* states
- EM waves carry electromagnetic energy

Mathematical Description of EM Waves

Start with an oscillating electric field in $\hat{\mathbf{x}}$ direction:

$$\mathbf{E} = E_0 e^{i(kz-\omega t)}\hat{\mathbf{x}} \qquad c = \frac{\omega}{k}$$

This is known as a plane wave solution

The direction of propagation of the wave is $\hat{\mathbf{z}}$

Use M4 to determine magnetic field:

$$\nabla \times \mathbf{B} = \left[\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right]\hat{\mathbf{x}} = \epsilon_0 \mu_0 \frac{\partial E_x}{\partial t}\hat{\mathbf{x}}$$

$$\mathbf{B} = B_0 e^{i(kz-\omega t)}\hat{\mathbf{y}} \qquad B_0 = \frac{E_0}{c}$$

E and **B** are perpendicular to each other, and to the direction of propagation of the wave

E and **B** have amplitude ratio c, and are *in phase*

Polarization of EM Waves

For an EM wave propagating in the $\hat{\mathbf{z}}$ direction there are two independent polarization states

These can be defined in various ways:

- **Plane polarization:**
  States are (E_x, B_y) and (E_y, B_x), i.e. the directions of the fields are independent of z and t

- **Circular polarization:**
  The directions of $\mathbf{E}$ and $\mathbf{B}$ rotate about the z axis as a function of t. Sense of rotation is anticlockwise for one state, and clockwise for the other (known as left and right-handed).

An **unpolarized** wave is a *random* mix of the two states

A **fully polarised** wave contains only one state

Notes:

Diagrams:

Energy of EM Waves

The total energy stored in the oscillating fields is:

$$U = U_E + U_M = \frac{1}{2}\int_V \left(\frac{|\mathbf{B}|^2}{\mu_0} + \epsilon_0|\mathbf{E}|^2\right) d\tau$$

The **time averaged** energy density is:

$$\frac{d < U >}{d\tau} = \frac{1}{2}\epsilon_0 E_0^2 = \frac{1}{2}\frac{B_0^2}{\mu_0}$$

The electric and magnetic energy densities are equal

The **time variation** of the energy is:

$$\frac{\partial U}{\partial t} = \int_V \left(\frac{\mathbf{B}}{\mu_0}\frac{\partial \mathbf{B}}{\partial t} + \epsilon_0 \mathbf{E}\frac{\partial \mathbf{E}}{\partial t}\right) d\tau$$

Notes:

Diagrams:

Use M3 and M4 to replace time derivatives with space derivatives:

$$\frac{\partial B_y}{\partial t} = \frac{\partial E_x}{\partial z} \qquad \epsilon_0 \frac{\partial E_x}{\partial t} = -\frac{1}{\mu_0}\frac{\partial B_y}{\partial z}$$

$$\frac{\partial U}{\partial t} = \frac{1}{\mu_0}\int_V \left(B_y \frac{\partial E_x}{\partial z} - E_x \frac{\partial B_y}{\partial z} \right) d\tau$$

This can be written more generally as:

$$\frac{\partial U}{\partial t} = -\frac{1}{\mu_0}\int_V \frac{\partial}{\partial z}(\mathbf{E} \times \mathbf{B}) d\tau$$

The **Poynting vector** $\mathbf{N} = (\mathbf{E} \times \mathbf{B})/\mu_0$ describes the flow of electromagnetic energy along $\hat{\mathbf{z}}$ in units of W/m^2

Notes:

Diagrams: