As in S we have

$$e^{i\omega(t+(x/c)\cos\alpha+(y/c)\sin\alpha)}\,, \qquad \text{where} \quad \vec{k} = -\frac{\omega}{c}\,(\cos\alpha, \sin\alpha, 0)\ ,$$

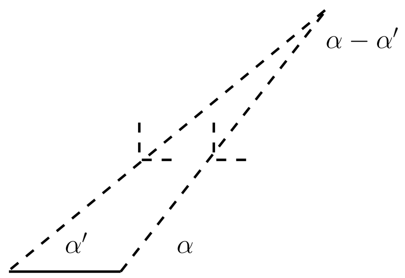
and so

$$\begin{aligned}
\tan(\alpha-\alpha') &= \frac{\tan\alpha - \tan\alpha'}{1+\tan\alpha\tan\alpha'} \\
&= \sin\alpha\frac{\gamma(\cos\alpha + v/c) - \cos\alpha}{\gamma\cos\alpha(\cos\alpha+v/c)+\sin^2\alpha} \\
&= \frac{v}{c}\sin\alpha\frac{1+v/2c\ \cos\alpha + O(v^2/c^2)}{1+v/c\ \cos\alpha+O(v^2/c^2)}
\end{aligned}$$

or

$$\boxed{\tan(\alpha-\alpha') = \frac{v}{c}\sin\alpha\left[1-\frac{v}{2c}\cos\alpha + O\left(\frac{v^2}{c^2}\right)\right]}\,.$$

Non-relativistic (Bradley)



As $\tan(\alpha-\alpha') \approx \alpha - \alpha'$ then

$$\alpha - \alpha' = \frac{v}{c}\sin\alpha\,.$$

7.4 Minkowski (Space–Time) diagrams

Portray x, y, z, t as a point in a four-dimensional space-time: $(ct, \vec{r}) \equiv (ct, x, y, z)$.

- Every point P represents an *event* in space-time
- Under a Lorentz transformation for $S \to S'$

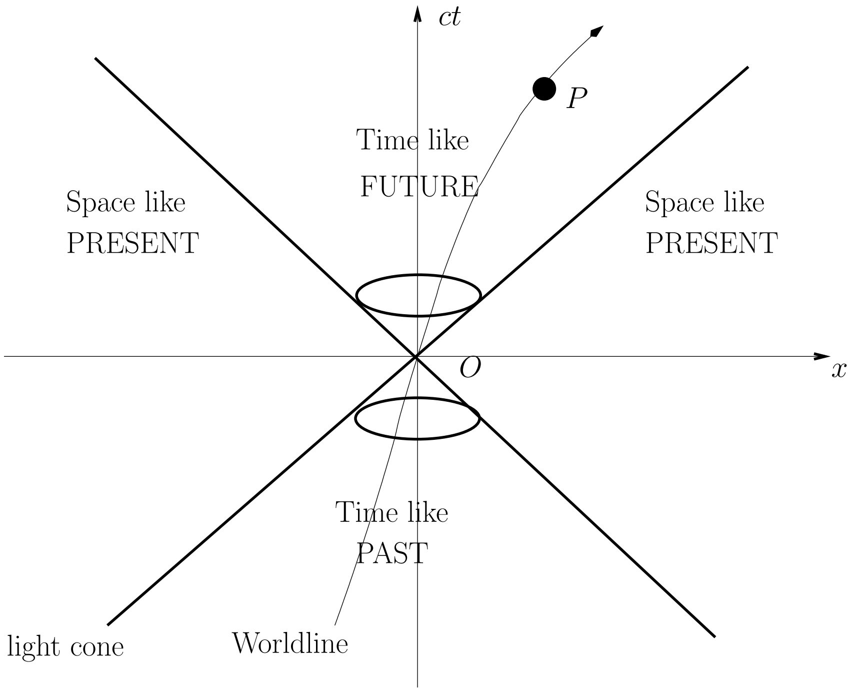
$$s^2 = c^2t^2 - \vec{r}^2\,, \qquad \vec{r}^2 = x^2+y^2+z^2\,,$$

is a Lorentz invariant, ie $s^2 = s'^2$ (after a few lines of algebra). So

$$s^2 = \begin{cases} > 0 & \text{time-like} \\ = 0 & \text{light-like} \\ < 0 & \text{space-like} \end{cases}$$

This decomposition remains the same in all inertial frames (ie after any Lorentz transformation)

Minkowski or Space–time picture (wlog two dimensional)



- Light signal define a **cone** (at 45° as choose ct for y-axis)
- All time-like points lie *within* light-cone
- All space-like points lie *outside* light-cone
- World-lines are motion of a particle (starting at $t = 0$, $x = 0$). World-lines of a photon lie *on* the light cone

Consider two space-time points $P_1(ct_1, \vec{r}_1)$ and $P_2(ct_2, \vec{r}_2)$. Then

$$s_{12}^2 = c^2(t_1 - t_2) - |\vec{r}_1 - \vec{r}_2|^2\,,$$

also a Lorentz invariant.

Space-like separation

From $s_{12}^2 < 0$, then $x_1 - x_2 > c(t_1 - t_2)$, ie events not connected by a light signal

no causal connection

Can find a Lorentz transformation to S' where both events P_1 and P_2 are at the same time, as

$$c(t_1' - t_2') = \gamma \left(c(t_1 - t_2) - \frac{v}{c}(x_1 - x_2) \right) .$$

Choose

$$v = c\frac{c(t_1 - t_2)}{x_1 - x_2} < c \,, \qquad \text{so } S' \text{ possible}$$

Hence can always find Lorentz transformations so that the order of space-like events is interchanged.

Time-like separation

From $s_{12}^2 > 0$, then $c(t_1 - t_2) > x_1 - x_2$, ie events connected by a light signal

causal connection possible

Cannot find a Lorentz transformation to S' where both events P_1 and P_2 are at the same time, as would need

$$v = c\frac{c(t_1 - t_2)}{x_1 - x_2} > c \,, \qquad \text{so } S' \text{ not possible}$$

So cannot interchange cause and effect. However as

$$x_1' - x_2' = \gamma \left((x_1 - x_2) - v(t_1 - t_2) \right) .$$

can find a frame S' where $x_1' = x_2'$, events happen at the same place

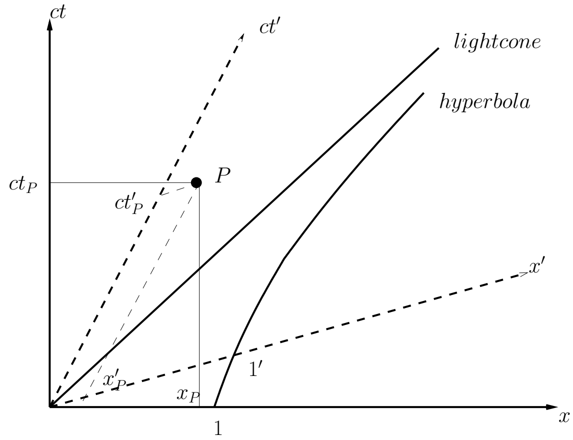
Define **proper time** by

$$\boxed{c^2\tau^2 = s^2}\,, \qquad \text{or} \quad \tau = \sqrt{t^2 - \frac{\vec{r}^2}{c^2}} \,.$$

So here we have

$$\tau_{P_1} - \tau_{P_2} = \sqrt{(t_1 - t_2)^2 - \frac{|\vec{r}_1 - \vec{r}_2|^2}{c^2}} \,.$$

ie time measured on a clock moving with uniform speed $|vecr_1 - \vec{r}_2|/(t_1 - t_2)$ between the two events, P_1 and P_2.



7.4.1 Diagramatic relation between S and S'

We now briefly consider the relationship between S and S' on a Minkowski diagram

Co-ordinate systems same at $t = 0 = t'$

- S' time-axis defined by $x' = 0 = \gamma(x - vt)$ or

$$ct = \frac{1}{v/c}\, x\,,$$

  ie straight line, gradient > 1, ie between S time-axis and light signal

- S' space-axis defined by $t' = 0 = \gamma(t - v/c^2 x)$ or

$$ct = v/c\, x\,,$$

  ie straight line, gradient < 1, ie between S space-axis and light signal

Scaling of axes

The Lorentz transformation $S \to S'$ changes the scale of the axes.

Choose a length unit by $\bar{s}^2 = -1$. Then

$$x^2 = (ct)^2 + 1\,,$$

– ie **hyperbola**, all points on it have length $\bar{s}^2 = -1$. In particular $t = 0$ cuts x-axis at $x = 1$.

But as s^2 is a Lorentz invariant, then this is equivalent to

$$x'^2 = (ct')^2 + 1\,,$$