giving

$$t = \frac{mc}{eE} \sinh\left(\frac{eE}{mc}\tau\right) .$$

As $\sinh x > x$ then $t > \tau$, ie time dilation.

In this case the $\mu = 0$ component yields no new information, but is consistent with the above results. For

$$\begin{aligned} \text{LHS} &= m\frac{d^2x^0}{d\tau^2} = m\gamma\frac{d}{dt}(\gamma c) \\ \text{RHS} &= eF^{01}\frac{dx_1}{d\tau} = e\left(-\frac{1}{c}E)\right)(-\gamma\frac{dx}{dt}) = e\gamma\frac{E}{c}\frac{dx}{dt} , \end{aligned}$$

giving

$$\frac{d\gamma}{dt} = \frac{1}{c}\frac{eE}{mc}\frac{dx}{dt} , \qquad \Longrightarrow \qquad \gamma(u) = \frac{1}{c}\frac{eE}{mc}x + 1 .$$

So

$$x(t) = c\frac{mc}{eE}(\gamma - 1) = c\frac{mc}{eE}\left(\sqrt{1 + \left(\frac{eE}{mc}\right)^2 t^2} - 1\right) ,$$

agreeing with previous result.

10.2 Understanding (some of) the equations on the floor outside the JCMB 'Magnet' Café

Let us now try to re-write Maxwell's equations using $F^{\mu\nu}$.

10.2.1 Maxwell's equations

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\ \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{B} &= \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{aligned}$$

(much more useful to scale the electric and magnetic fields,

$$\vec{E} = \sqrt{\epsilon_0}\vec{E}_{SI} , \qquad \vec{B} = c\sqrt{\epsilon_0}\vec{B}_{SI} \equiv \frac{1}{\sqrt{\mu_0}}\vec{B}_{SI} , \qquad e = \frac{1}{\sqrt{\epsilon_0}}e_{SI} , \quad \vec{j} = \frac{1}{\sqrt{\epsilon_0}}\vec{j}_{SI} ,$$

Heaviside–Lorentz units $\sim$ Gaussian units without 4π)

To try to handle the SI ghastly unit problem compromise and re-write

$$\begin{aligned}\vec{\nabla}\cdot\vec{E} &= c\,\mu_0\rho c \\ \vec{\nabla}\times\vec{E}+\frac{\partial c\vec{B}}{\partial ct} &= 0 \\ \vec{\nabla}\cdot c\vec{B} &= 0 \\ \vec{\nabla}\times c\vec{B} &= c\,\mu_0\vec{j}+\frac{\partial\vec{E}}{\partial ct}\end{aligned}$$

Try to re-write them in a covariant (ie form invariant) way using $F^{\mu\nu}$. Expect the form to be $\partial_\mu = (\partial/\partial ct, \vec{\nabla})_\mu$

$$\partial_\mu F^{\mu\nu} = \ldots .$$

$\nu = 0$

$$\partial_\mu F^{\mu 0} = \partial_i F^{i0} = \frac{1}{c}\vec{\nabla}\cdot\vec{E} = \mu_0\rho c\,.$$

$\nu = 3$

$$\begin{aligned}\partial_\mu F^{\mu 3} &= \partial_0 F^{03} + \partial_i F^{i3} \\ &= \frac{1}{c}\left(-\frac{\partial E^3}{\partial ct}\right) + \frac{1}{c}\frac{\partial}{\partial x^1}(cB^2) + \frac{1}{c}\frac{\partial}{\partial x^2}(-cB^1) \\ &= \frac{1}{c}\left[\vec{\nabla}\times(c\vec{B}) - \frac{\partial\vec{E}}{\partial ct}\right]^3 = \mu_0 j^3\,.\end{aligned}$$

(Similarly for the other components.)

Hence we have

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu\,, \quad \text{with} \quad \boxed{J^\mu = (\rho c, \vec{j})^\mu}\,.$$

This represents two of Maxwell's equations - the 'source' equations. Note that J^μ must be a vector as the LHS is.

For the other two equations we note that they are the 'source free' equations and may be obtained from the two source equations by setting the sources equal to zero and 'interchanging' the electric and magnetic fields,

$$\vec{E}\to c\vec{B}\,, \qquad \text{and} \quad c\vec{B}\to -\vec{E}\,.$$

So define a new second field strength tensor, the antisymmetric 'dual' field strength tensor by this rotation

$$\boxed{{}^*F^{\mu\nu} = \frac{1}{c}\begin{pmatrix} 0 & -cB^1 & -cB^2 & -cB^3 \\ cB^1 & 0 & E^3 & -E^2 \\ cB^2 & -E^3 & 0 & -E^1 \\ cB^3 & E^2 & -E^1 & 0 \end{pmatrix}},$$

(on the JCMB floor it is called $G^{\mu\nu}$).

Thus the four Maxwell's equations become

$$\boxed{\partial_\mu F^{\mu\nu} = \mu_0 J^\nu}, \quad \text{and} \quad \boxed{\partial_\mu {}^*F^{\mu\nu} = 0}.$$

- In distinction to Newtonian mechanics no modifications of Maxwell's equations are necessary - they are already relativistic
- A immediate trivial consequence is

  $$\partial_\mu \partial_\nu F^{\mu\nu} = 0 \qquad \Longrightarrow \qquad \partial_\mu J^\mu = 0\,.$$

  This is just

  $$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0\,,$$

  ie current conservation (see JCMB floor).
- If we define the 4-dimensional Levi-Cevita (pseudo) tensor

  $$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of } 0123 \\ -1 & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of } 0123 \\ 0 & \text{otherwise} \end{cases}$$

  then we can write

  $$\boxed{{}^*F^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}}.$$

  (For example ${}^*F^{01} = \frac{1}{2}\epsilon^{01\rho\sigma}F_{\rho\sigma} = \frac{1}{2}(F_{23} - F_{32}) = -cB^1/c$.)
- This means that the dual field strength is a pseudotensor.

10.2.2 Lorentz transformations of the electric and magnetic fields

As $F^{\mu\nu}$ is a tensor then we can make a Lorentz transformation (in the x-direction),

$$F'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}\,.$$

This gives the Lorentz transformed electric and magnetic fields

$$\boxed{\begin{array}{rcl} E'_x &=& E_x \\ E'_y &=& \gamma(E_y - vB_z) \\ E'_z &=& \gamma(E_z + vB_y) \end{array}} \quad \text{and} \quad \boxed{\begin{array}{rcl} B'_x &=& B_x \\ B'_y &=& \gamma(E_y + v/c^2 E_z) \\ B'_z &=& \gamma(B_z - v/c^2 E_y) \end{array}}$$

For example

$$\begin{aligned} -\frac{1}{c}E'_x \equiv -\frac{1}{c}E'^1 &= \Lambda^0{}_\rho \Lambda^1{}_\sigma F^{\rho\sigma} \\ &= \Lambda^0{}_0\Lambda^1{}_0 F^{00} + \Lambda^0{}_0\Lambda^1{}_1 F^{01} + \Lambda^1{}_0\Lambda^0{}_1 F^{10} + \Lambda^1{}_1\Lambda^0{}_1 F^{11} \\ &= 0 + (\gamma)(\gamma)(-E_x/c) + (-\gamma v/c)(-\gamma v/c)(E_x/c) + 0 \\ &= \frac{1}{c}\gamma^2\left(\frac{v^2}{c^2} - 1\right)E_x = -\frac{1}{c}E_x\,, \end{aligned}$$

and similarly for the other components (messy). Alternatively could also write in matrix form as $F'^{\mu\nu} = (\Lambda F \Lambda^T)^{\mu\nu}$ (still messy).

So a purely electric or magnetic field in one frame will appear as a mixture in another frame (there are constraints, see section 10.2.3). So really have one electromagnetic field $F^{\mu\nu}$ rather than separate $\vec{E}$, $\vec{B}$.

Can easily generalise this to the general case. Consider $\parallel$, $\perp$ components to $\vec{v}$. We also have

$$\vec{v} \times \vec{B} = -vB_z\vec{e}_y + vB_y\vec{e}_z\,,$$

which gives $E'_y = \gamma(E_y + (\vec{v}\times\vec{B}_\perp)_y)$, $E'_z = \gamma(E_z + (\vec{v}\times\vec{B}_\perp)_z)$ (and similarly for B'_y, B'_z). So

$$\boxed{\begin{array}{rcl} \vec{E}'_\parallel &=& \vec{E}_\parallel \\ \vec{E}'_\perp &=& \gamma(\vec{E}_\perp + \vec{v}\times\vec{B}_\perp) \end{array}} \quad \text{and} \quad \boxed{\begin{array}{rcl} \vec{B}'_\parallel &=& \vec{B}_\parallel \\ \vec{B}'_\perp &=& \gamma(\vec{B}_\perp - \frac{1}{c^2}\vec{v}\times\vec{E}_\perp) \end{array}}$$

Alternatively using $\vec{E} = \vec{E}_\parallel + \vec{E}_\perp$ and $\vec{E}_\parallel = (\vec{v}\cdot\vec{E})/v^2\,\vec{v}$ this gives

$$\begin{aligned} \vec{E}' &= \gamma(\vec{E} + \vec{v}\times\vec{B}) + (1-\gamma)\frac{\vec{v}\cdot\vec{E}}{v^2}\vec{v} \\ \vec{B}' &= \gamma(\vec{B} - \frac{1}{c^2}\vec{v}\times\vec{E}) + (1-\gamma)\frac{\vec{v}\cdot\vec{B}}{v^2}\vec{v} \end{aligned}$$

10.2.3 Invariants

We know that $F^{\mu\nu}$, ${}^*F^{\mu\nu}$ and $\eta^{\mu\nu}$ are tensors so we can form (invariant) scalars from them. The obvious one is $F^{\mu\nu}\eta_{\mu\nu}$ but this is trivial $(= 0)$.

(a) Using just $F^{\mu\nu}$,

$$\boxed{F_{\mu\nu}F^{\mu\nu} = \frac{2}{c^2}\left((c\vec{B})^2 - \vec{E}^2\right)},$$

as

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= F_{0\nu}F^{0\nu} + F_{1\nu}F^{1\nu} + F_{2\nu}F^{2\nu} + F_{3\nu}F^{3\nu} \\ &= -F^{0j}F^{0j} - F^{10}F^{10} + F^{12}F^{12} + F^{13}F^{13} + \ldots \\ &= \frac{1}{c^2}\left[-\vec{E}^2 - E_x^2 + c^2(B_z^2 + B_y^2) - E_y^2 + c^2(B_x^2 + B_z^2) - E_z^2 + c^2(B_y^2 + B_x^2)\right] \\ &= \frac{2}{c^2}\left((c\vec{B})^2 - \vec{E}^2\right) . \end{aligned}$$

Hence the difference in energy densities $\frac{1}{2}(\epsilon_0\vec{E}^2 - 1/\mu_0\vec{B}^2)$ is an invariant.

(b) Using both $F^{\mu\nu}$ and ${}^*F^{\mu\nu}$,

$$\boxed{F_{\mu\nu}{}^*F^{\mu\nu} = -\frac{4}{c}\vec{E}\cdot\vec{B}}.$$

(The proof is the same as for (a).)

Hence $\vec{E}\cdot\vec{B}$ is an invariant.

Note that a frame S' can be chosen in which $\vec{E}'$ or $\vec{B}'$ is zero only if $\vec{E} \perp \vec{B}$ in S. (If $\vec{E}\cdot\vec{B} = \vec{E}'\cdot\vec{B}' = 0$ say then $\vec{E}\cdot\vec{B} = 0$.)

$E^2 > c^2B^2$ we can choose S' so that $\vec{B}' = 0$

$E^2 < c^2B^2$ we can choose S' so that $\vec{E}' = 0$

(ie $c^2B^2 - E^2 = c^2B'^2 - E'^2$. So frame with $\vec{B}' = 0$ gives $E^2 > c^2B^2$ and $\vec{E}' = 0$ gives $E^2 < c^2B^2$.)

10.2.4 Example - the Coulomb field

Consider in S a point charge q moving with velocity $\vec{v} = (v, 0, 0)$. The electric field of the charge at rest in S' is

$$\vec{E}'(\vec{r}') = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}'}{r'^3}, \qquad \vec{B}' = 0 .$$

So in S we have

$$\begin{aligned}
E_x = E'_x &= \frac{q}{4\pi\epsilon_0}\frac{\gamma(x-vt)}{(\gamma^2(x-vt)^2+y^2+z^2)^{3/2}} \\
E_y = \gamma E'_y &= \frac{q}{4\pi\epsilon_0}\frac{\gamma y}{(\gamma^2(x-vt)^2+y^2+z^2)^{3/2}} \\
E_z = \gamma E'_z &= \frac{q}{4\pi\epsilon_0}\frac{\gamma z}{(\gamma^2(x-vt)^2+y^2+z^2)^{3/2}}
\end{aligned}$$

(inverse Lorentz transformation for fields and re-writing x', y', z' from the Lorentz transfomation as functions of x, y, z).

Field in S is as in S' a central field, but in S it is not isotropic, due to γ^2 factor.

For the magnetic field in S we have $\vec{B}_\parallel = \vec{B}'_\parallel = 0$, $\vec{B}_\perp = \gamma(\vec{B}'_\perp + \vec{v}\times\vec{E}'/c^2) = \gamma\vec{v}\times\vec{E}'/c^2 = \vec{v}\times\vec{E}/c^2$ or

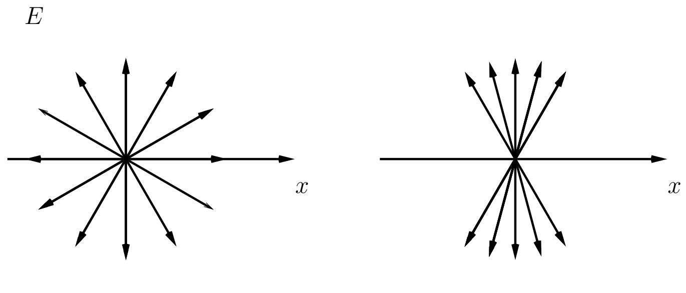
$$\vec{B} = \frac{1}{c^2}\,\vec{v}\times\vec{E}\,.$$

So the observer there sees a moving charge, ie current, and magnetic field even in the non-relativistic limit $v \ll c$,

$$\vec{B} \to \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\vec{r}}{|\vec{r}-\vec{v}t|^3}\,,$$

which is essentially the Biot-Savart law.

Consider the ultra-relativistic limit $v \to c$



stationary *moving*

- Near the x-axis ($y \sim 0$, $z \sim 0$, $x - vt \neq 0$) then

$$E_x \approx \frac{1}{\gamma^2}\frac{q}{4\pi\epsilon_0}\frac{1}{(x-vt)^2}\,,$$

  ie a reduction in the field strength by factor γ^2 compared to the static case

- In the y-z plane (through q) then

$$E_y = \gamma \frac{q}{4\pi\epsilon_0} \frac{y}{(y^2+z^2)^{3/2}}, \qquad E_z = \gamma \frac{q}{4\pi\epsilon_0} \frac{z}{(y^2+z^2)^{3/2}},$$

  ie an increase in the field strength by a factor γ

- As $\vec{E}$ is $\perp$ to $\vec{B}$ then the $\vec{B}$ field lines are circles in the y-z plane about q. (Note the invariant $\vec{E} \cdot \vec{B}$ remains 0 as it should.)

The End