Symmetries of Classical Mechanics (SoCM) – Example sheet III: 14.10.11

Semester 1 2011

1. The mechanical force per unit volume acting on a conductor carrying a steady current density $\underline{j}$ is given by $\underline{f} = \underline{j} \times \underline{B}$. Using the relevant Maxwell's equations namely $\underline{\nabla} \cdot \underline{B} = 0$ and $\underline{\nabla} \times \underline{H} = \underline{j}$ where $\underline{B} = \mu \underline{H}$ and taking μ to be constant show that

$$f_i = \frac{\partial s_{ij}}{\partial x_j} \quad \text{where} \quad s_{ij} = \mu \left(H_i H_j - \tfrac{1}{2} H_k^2 \delta_{ij} \right) .$$

   [s_{ij} is part of the Maxwell stress tensor.]

2. Three right-handed cartesian frames of reference S, S', S'', with common origin O, are related as follows: rotation of S about Oz through an angle θ brings S into coincidence with S', and a rotation of S' about Ox' through an angle ϕ brings S' into coincidence with S''. Find the components in the frame S'' of a vector $\underline{x}$ in terms of its components (x_1, x_2, x_3) in the frame S.

3. If the position vector $\underline{x}$ is rotated through an angle θ about the axis defined by the unit vector $\underline{n}$ show that it coincides with the vector $\underline{y}$ given by

$$\underline{y} = \underline{x} \cos\theta + (\underline{x} \cdot \underline{n}) \underline{n} (1 - \cos\theta) - \underline{x} \times \underline{n} \sin\theta .$$

   If this is written in the form $y_i = R_{ij}(\theta, \underline{n}) x_j$ find the elements of R_{ij} and show that

$$\begin{aligned} R_{ii} &= 1 + 2\cos\theta , \\ \epsilon_{ijk} R_{jk} &= -2 n_i \sin\theta . \end{aligned}$$

   Show that $RR^T = R^T R = I$ by evaluating $R_{ik} R_{jk}$ etc..
   Verify that $\underline{n}$ is an eigenvector of R with eigenvalue 1. By considering the orthonormal triad $(\underline{u}, \underline{v}, \underline{n})$, show that $\underline{u} \pm i\underline{v}$ are eigenvectors with eigenvalues $\exp(\mp i\theta)$.

4. Find the eigenvalues, λ, and eigenvectors, $\underline{e}$, of the following matrices:

$$M = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 4 & -2 \\ -1 & -2 & 2 \end{pmatrix} , \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix} .$$

   What can one say about the eigenvectors?

5. A rotation of magnitude θ about a unit vector $\underline{n}$ is given by $y_i = R_{ij}x_j$, with

$$R = \begin{pmatrix} 4/9 & 1/9 & 8/9 \\ 7/9 & 4/9 & -4/9 \\ -4/9 & 8/9 & 1/9 \end{pmatrix}.$$

Find the angle θ and axis $\underline{n}$ for this rotation. Two identical successive rotations are described by the rotation matrix $S_{ij} = R_{ik}R_{kj}$. Find the angle and axis of the rotation associated with S.