Symmetries of Classical Mechanics (SoCM) – Example sheet I: 30.09.11

Semester 1 2011-2012

1. To what quantities do the following expressions in suffix notation (with summation convention) correspond?

$$\delta_{ii}\,,\quad \delta_{ij}a_ia_j\,,\quad \delta_{ij}\delta_{ij}\,,\quad \epsilon_{iji}\,,\quad \epsilon_{ijk}\delta_{ij}\,,\quad b_i\epsilon_{ijk}a_kc_j\,,\quad \epsilon_{ijk}a_{3i}a_{1k}a_{2j}\,.$$

2. Verify the identity

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\,,$$

for the cases

(a) $i = 1$, $j = 2$, $l = 1$, $m = 2$

(b) $i = 1$, $j = 2$, $l = 2$, $m = 1$

(c) $i = 1$, $j = 2$

3. Evaluate $\epsilon_{ijk}\delta_{jk}$.

Solve the following equation for v_k, f, p_{ij} in terms of k_{ij}:

$$\epsilon_{ijk}v_k + \delta_{ij}f + p_{ij} = k_{ij}\,,$$

where $p_{ij} = p_{ji}$ and $p_{ii} = 0$.

4. (a) Write in suffix notation the vector equation

$$\underline{a}\times\underline{b} + \underline{c} = (\underline{a}\cdot\underline{b})\underline{b} - \underline{d}\,.$$

(b) If u_i ($i = 1, 2, 3$) are three independent variables show that

$$\frac{\partial u_i}{\partial u_j} = \delta_{ij}\,.$$

(c) Show both by using the identity $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ and also by writing the indices explicitly that

$$\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl}\,,\qquad \epsilon_{ijk}\epsilon_{ijk} = 6\,.$$

5. Prove, by writing out the components explicitly, that

$$\underline{a}\times(\underline{b}\times\underline{c}) = (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c}\,,$$

where $\underline{a}$, $\underline{b}$ and $\underline{c}$ are any three vectors. Use this to deduce a relationship for $\epsilon_{ijk}\epsilon_{klm}$ in terms of the Kronecker delta.
Find expressions for $(\underline{l} \times \underline{m}) \cdot (\underline{l} \times \underline{n})$ and $(\underline{l} \times \underline{m}) \times (\underline{l} \times \underline{n})$ where $\underline{l}$, $\underline{m}$, $\underline{n}$ are any three vectors.
By taking $\underline{l}$, $\underline{m}$, $\underline{n}$ to be vectors from the centre of a sphere to points on its surface, deduce the sine and cosine formulae for spherical triangles,

$$\begin{aligned} \cos(a) &= \cos(b)\cos(c) + \sin(b)\sin(c)\cos(A)\,, \\ \frac{\sin(a)}{\sin(A)} &= \frac{\sin(b)}{\sin(B)} = \frac{\sin(c)}{\sin(C)}\,, \end{aligned}$$

where a, b, c are the angles subtended by the radii at the centre of the sphere and A, B, C are the angles on the surface, with A opposite the arc subtending a, etc..

6. Explain why any four vectors $\underline{u}$, $\underline{v}$, $\underline{w}$, $\underline{z}$, in three dimensional (Euclidean) space obey a relationship of the form

$$a\underline{u} + b\underline{v} + c\underline{w} + d\underline{z} = 0\,.$$

The position vectors relative to the origin O of the vertices A, B, C, D of a tetrahedron are $\underline{a}$, $\underline{b}$, $\underline{c}$, $\underline{d}$. (O lies inside the tetrahedron.) Show that the equation for the plane BCD can be written as

$$\underline{r} = \lambda\underline{b} + (1 - \lambda - \mu)\underline{c} + \mu\underline{d}\,,$$

and find the point of intersection of OA with this plane. AO, BO, CO, DO meet the opposite faces of the tetrahedron in E, F, G, H respectively. Show using vectors that

$$\frac{AO}{AE} + \frac{BO}{BF} + \frac{CO}{CG} + \frac{DO}{DH} = 3\,.$$