Symmetries of Classical Mechanics (SoCM) – Example sheet VII: 11.11.11

Semester 1 2011-2012

1. Evaluate the integrals

   (a)

   $$I = \int_{\text{all space}} e^{-a_{ij}x_i x_j}\, dV\ ,$$

   where a_{ij} are the elements of a real symmetric positive definite matrix.
   [It may be assumed without proof that $\int_{-\infty}^{\infty} d\xi e^{-a\xi^2} = \sqrt{\frac{\pi}{a}}$ for $\mathrm{Re}(a) > 0$.]
   [Hint: rotate to the principal axis basis.]

   (b)

   $$A_{ij} = \int_{\text{sphere}\, r\leq a} x_i x_j\, dV\ .$$

   [Hint: first argue that A_{ij} is a rank two isotropic tensor and then, for example, consider the sum over $i = j$.]

   (c)

   $$A_{ijkl} = \int_{\text{sphere}\, r\leq a} x_i x_j x_k x_l\, dV\ .$$

   [Hint: first argue that A_{ijkl} is a rank four isotropic tensor and then proceed as in the previous part.]

2. A piece of material is subjected to a homogeneous stress such that the forces per unit area across faces with (orthogonal) unit normals $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$ are $(5p\underline{e}_1 + 2p\underline{e}_3)$, $p\underline{e}_2$ and $(2p\underline{e}_1 + 2p\underline{e}_3)$ respectively. Construct the (homogeneous) stress tensor within the material and calculate the force acting across the element of area δS whose normal is in the direction $(\underline{e}_1 + 2\underline{e}_2 + 2\underline{e}_3)$.
   Find the principal axes of stress and the corresponding principal stresses.

3. If the stress-strain relation is given by

   $$P_{ij} = \lambda \mathrm{Tr} E \delta_{ij} + 2\mu E_{ij}\ ,$$

   then by considering a simple tension/unit area S in an axially symmetric rod placed in the x_1 direction, show that this may alternatively be written as

   $$P_{ij} = \frac{Y}{(1+\sigma)}\left[\frac{\sigma}{(1-2\sigma)}\mathrm{Tr} E \delta_{ij} + E_{ij}\right]\ ,$$

   where Y is Young's modulus, defined by $Y = S/E_{11}$ and σ is Poisson's ratio, defined by $\sigma = -E_{22}/E_{11} \equiv -E_{33}/E_{11}$.
   [Hint: look at your notes.]

4. The point (x_1, x_2, x_3) in an isotropic, homogeneous material is displaced by a small amount (u_1, u_2, u_3) given by

$$u_1 = -\frac{\sigma}{Y}\alpha\, x_1 + \frac{(1+\sigma)}{Y}\beta\, x_2\,, \quad u_2 = \frac{(1+\sigma)}{Y}\beta\, x_1 - \frac{\sigma}{Y}\alpha\, x_2\,, \quad u_3 = \frac{1}{Y}\alpha\, x_3\,,$$

where Y and σ are Young's modulus and Poisson's ratio respectively, and α and β are constants. Evaluate the strain tensor, and hence find the nature and magnitude of the stresses that are causing this deformation.

5. The point (x_1, x_2, x_3) in an isotropic, homogeneous medium undergoes a small displacement (u_1, u_2, u_3) given by

$$u_1 = \alpha x_1\,, \qquad u_2 = -3d\, x_2 - d\, x_3\,, \qquad u_3 = -d\, x_2 - 3d\, x_3\,,$$

where α and d are constants. Write down the strain tensor and find the principal strains β and γ ($|\beta| < |\gamma|$) in the $\underline{e}_2$-$\underline{e}_3$ plane.
Show that this strain may be maintained by a stress P_1 along the $\underline{e}_1$ axis, a stress P_2 along the principal axis of strain which has principal strain β, and zero stress along the third principal axis, provided that $d = \sigma\alpha/(4-2\sigma)$, where σ is Poisson's ratio for the medium.

6. A bar of homogeneous, isotropic material has a rectangular cross-section. It is subjected to a uniform, normal pressure P over its four sides, and the end faces are subjected to a tension αP per unit area acting parallel to the axis of the bar. Find the ratio of the lateral strain to the longitudinal strain in terms of α and Poisson's ratio, σ, for the material of the bar. Show that this ratio is independent of σ for $\alpha = 2$ or $\alpha = -1$ and state the corresponding values of the ratio.

7. (a) If $\underline{u}$ is a vector field, show that $\underline{\nabla} \cdot \underline{u}$ is a scalar field.

   (b) If ϕ is a scalar field, show that the quantity

$$T_{ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j}\,,$$

   is a second rank tensor.