Symmetries of Classical Mechanics (SoCM) – Example sheet VI: 04.11.11

Semester 1 2011-2012

1. The moments of inertia along the principal axes $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ of a rigid body are 1, 2 and 3 respectively. (The off-diagonal components of the inertia tensor are zero in this frame.) An observer has a coordinate basis $\underline{e}'_1 = \frac{1}{2}(\underline{e}_1 + \sqrt{3}\underline{e}_3)$, $\underline{e}'_2 = \underline{e}_2$ and $\underline{e}'_3 = \frac{1}{2}(\underline{e}_3 - \sqrt{3}\underline{e}_1)$. Confirm that this is an orthonormal basis, and find the values of the components of the inertia tensor measured by the observer.
   In a third frame with axes $\{\underline{e}''_1, \underline{e}''_2, \underline{e}''_3\}$, $\underline{e}_1$ has components $\frac{1}{\sqrt{2}}(1, 1, 0)$ relative to those axes and $\underline{e}_2$ has components $\frac{1}{\sqrt{3}}(1, -1, 1)$. Find the inertia tensor I''_{ij} in this frame, and calculate $I''_{ij}I''_{ji}$ (there is a simple way to do this).

2. The electrical conductivity σ in a crystal is measured by an observer to have components

$$\begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

   Show that there is one direction in which no current flows, and find that direction. [Hint: diagonalise.]
   The rate of energy dissipation per unit volume is given by $\underline{E} \cdot \underline{j}$ where $\underline{E}$ is the electric field and $\underline{j}$ the resulting current density. For a fixed value of $|\underline{E}|^2$, find the minimum and maximum energy dissipation rates.

3. Prove directly (ie using the transformation law $T'_{ij} = l_{i\alpha} l_{j\beta} T_{\alpha\beta}$) that the quantities

$$\alpha = T_{ii}\,, \qquad \beta = T_{ij}T_{ji}\,, \qquad \gamma = T_{ij}T_{jk}T_{ki}\,,$$

   are invariant under rotation of the frame of reference.
   If T_{ij} is a 3×3 symmetric tensor, express these invariants in terms of its eigenvalues. Deduce that the cubic equation for the eigenvalues λ is

$$\lambda^3 - \alpha\lambda^2 + \tfrac{1}{2}(\alpha^2 - \beta)\lambda - \tfrac{1}{6}(\alpha^3 - 3\alpha\beta + 2\gamma) = 0\,.$$

4. The point (x_1, x_2, x_3) in a body is displaced by an amount (u_1, u_2, u_3),

$$\begin{aligned} u_1 &= \gamma(4x_1 - 3x_2 + x_3) + c_1 \\ u_2 &= \gamma(x_1 + 5x_2 + x_3) + c_2 \\ u_3 &= \gamma(-x_1 + x_2 + 4x_3) + c_3 \end{aligned}$$

with $|\gamma| \ll 1$ and $(c_1,\, c_2,\, c_3)$ a constant vector. Find the strain and rotation tensors describing this transformation. Find the angle and axis of the rotation and the principal axes of strain (eigenvectors of the pure strain tensor).

5. The point P in a body is displaced by an amount $\underline{u} = \epsilon\, \underline{a}(\underline{b} \cdot \underline{r})$, where $\underline{r}$ is the position vector of the point P relative to a fixed origin, $\underline{a}$ and $\underline{b}$ are constant unit vectors inclined at an angle θ to each other and ϵ is a small constant. Find the strain and the rotation tensors.
Deduce that the body has undergone a small rotation through an angle $(\frac{1}{2}\, \epsilon\, \sin\theta)$ about the normal to the plane containing $\underline{a}$ and $\underline{b}$, (i.e. show that the rotation vector is $\frac{1}{2}\epsilon\, (\underline{b} \times \underline{a})$), together with a small strain, one of whose principal axes is this normal, the others being the internal and external bisectors of the angle formed by $\underline{a}$ and $\underline{b}$ (i.e. $\frac{1}{\sqrt{2}}(\underline{a}+\underline{b})$ and $\frac{1}{\sqrt{2}}(\underline{a}-\underline{b})$). Show that the corresponding principal strains are 0 and $\frac{1}{2}\epsilon\{(\underline{a}\cdot\underline{b}) \pm 1\}$.

6. Calculate the inertia tensor of a uniform right circular cylinder of mass M, radius a and height $\sqrt{3}a$, about its centre of gravity.