Symmetries of Classical Mechanics (SoCM) – Example sheet X: 02.12.11

Semester 1 2011-2012

1. Show, by constructing suitable Lorentz transformations, that any vector a_μ can be transformed to one of the forms

$$\begin{array}{r}(a,0,0,0)\\(-a,0,0,0)\\(1,1,0,0)\\(-1,1,0,0)\\(0,a,0,0)\end{array}$$

   where $a > 0$.

2. Express the four-velocity $dx^\mu/d\tau$ and four-acceleration $d^2x^\mu/d\tau^2$ of a particle in terms of its three-velocity $\underline{u}$. Evaluate the invariants that can be formed from the two four-vectors without further differentiation.

   The rate of radiation energy, W, of a moving charged particle is an invariant. In the Lorentz frame in which the particle is instantaneously at rest $W = K\dot{\underline{u}}^2$ where K is a constant. What is the general expression for W?

3. (a) Two particles of mass m_1 and m_2 which have velocities $\underline{u}_1$, $\underline{u}_2$ as measured in some inertial reference frame coalesce to form a single particle of mass M and velocity $\underline{u}$. Write down the total 4-momentum of the initial and final states and deduce that

$$\begin{aligned}\underline{u} &= \frac{m_1\gamma(u_1)u_1 + m_2\gamma(u_2)u_2}{m_1\gamma(u_1) + m_2\gamma(u_2)}\\ M^2 &= m_1^2 + m_2^2 + 2m_1m_2\gamma(u_1)\gamma(u_2)(1 - \underline{u}_1\cdot\underline{u}_2/c^2)\,.\end{aligned}$$

   (b) What is the minimum mass M of a particle which can decay into two other particles of masses m_1, m_2?

4. A rocket ejects exhaust at a constant speed u relative to itself by a process that conserves 4-momentum (but not mass!). Let m be the mass of the rocket when the rocket has speed v in the initial rest frame of the rocket. By equating the instantaneous 4-momentum p with the 4-momentum $p + dp$ at a later time, show that

$$m\frac{dv}{dm} = -u\left(1 - \frac{v^2}{c^2}\right)$$

   Solve to find

$$v(t) = c\tanh\left(\frac{v}{c}\ln\frac{m(0)}{m(t)}\right)\,,$$

   and hence that $v \to c$ as $m(t) \to 0$.

5. (a) Rewrite the wave equation for a scalar field

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}\,,$$

   in terms of the coordinates

$$x_\pm = x \pm ct\,,$$

and hence show that the general solution is

$$\phi(x_+, x_-) = f(x_+) + g(x_-)\,,$$

where f, g are arbitrary (differentiable) functions.

(b) Show that under a Lorentz transformation

$$x'_\pm = \sqrt{\frac{c \mp v}{c \pm v}}\, x_\pm\,.$$

Hence show that if $\phi(x_+, x_-)$ is a solution of the wave equation then so is $\phi(x'_+, x'_-)$.

(c) Use these results to rederive the (relativistic) Doppler shift formula.

6. A long straight thin wire has constant charge density λ per unit length. find the electric and magnetic fields in a coordinate system moving with constant speed v in a direction parallel to the wire.

   Verify that $(c\underline{B})^2 - \underline{E}^2$ is an invariant.

   [Hint: The electric field of the stationary wire in cylindrical polar coordinates (ρ, θ, z) is $\underline{E} = \lambda/(2\pi\epsilon_0\rho)\underline{e}_\rho$.]

7. A particle of mass m and charge e moves in a constant magnetic field $\underline{B} = (0, 0, B)$. Show that

   (a) The (relativistic) energy is constant

   (b) Calculate the 3-momentum with the initial condition $\underline{u}_0 = (u_0, 0, 0)$

   (c) Show that the path of the particle with the initial condition $\underline{r}|_{t=0} = (0, \omega_0 u_0, 0)$ where $\omega_0 = eB/(m\gamma)$, $\gamma = \gamma(u_0)$ is

$$\underline{r}(t) = \frac{u_0}{\omega_0}\left(\sin\omega_0 t\,\underline{e}_x + \cos\omega_0 t\,\underline{e}_y\right)\,.$$

8. Show directly using the Lorentz transformed electric and magnetic fields that $(c\underline{B})^2 - \underline{E}^2$ is an invariant.

9. $F^{\mu\nu}$ is an antisymmetric tensor. Show that a transformation to a frame moving with velocity v along the x-axis (with $y' = y$, $z' = z$) leaves F^{01} invariant, ie $F'^{01} = F^{01}$.

   Two colliding beams of particles move parallel to the x-axis, one with density n_1 per unit volume and velocity u_1, the other with density n_2 and velocity u_2. The 4-vector describing a current of particles is given by $J^\mu = (nc, n\underline{u})^\mu$. By constructing a suitable antisymmetric tensor, show that $n_1 n_2 (u_1 + u_2)$ is invariant under Lorentz transformations in the x-direction.

10. $\underline{E}$ and $\underline{B}$ are uniform fields. By constructing a scalar and pseudoscalar, show that necessary conditions for the existence of an observer who sees the field as magnetic only are $\underline{E} \perp \underline{B}$, $|\underline{E}| < c|\underline{B}|$. By constructing a suitable Lorentz transformation show that these conditions are sufficient.