3.4 Two dimensional Hamiltonian systems

For Hamiltonian systems,

$$H(q,p) = \frac{p^2}{2m} + V(q) \equiv E \,,$$

$(q \equiv x_1,\, p \equiv x_2)$

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix} = \begin{pmatrix} \frac{p}{m} \\ -V'(q) \end{pmatrix} ,$$

so fixed points are given by

$$p^* = 0 \,, \qquad V'(q^*) = 0 \,.$$

Type of fixed point determined by eigenvalues of

$$A = \begin{pmatrix} 0 & \frac{1}{m} \\ -V''(q^*) & 0 \end{pmatrix} ,$$

where $\delta\dot{\vec{r}} = A\delta\vec{r}$. These are

$$\lambda = \pm\sqrt{-V''(q^*)/m} \qquad \left\{ \begin{array}{ll} V'' > 0 & \text{Pot. Minimum: Elliptic fixed points} \\ V'' < 0 & \text{Pot. Maximum: Hyperbolic fixed points} \end{array} \right\} \text{only} \,.$$

- For an elliptic fixed point then $\delta\dot{\vec{r}} = A\delta\vec{r}$ gives

  $$\delta\ddot{q} + (V''(q^*)/m)\, \delta q = 0 \,,$$

  ie simple harmonic oscillation with period

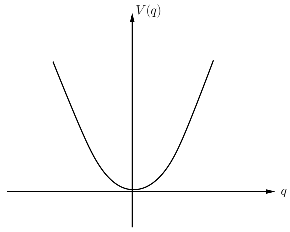
  $$T = \frac{2\pi}{\sqrt{V''(q^*)/m}} \,.$$

- A *separatrix* is a phase space trajectory which is a boundary between phase space trajectories with different behaviours (usually between (bounded) periodic and unbounded behaviours), ie it goes *through* a *hyperbolic point.* This allows a determination of $E_{separatrix}$ and hence the equation of the separatrix.

3.5 Examples of two dimensional systems

3.5.1 Simple Harmonic Oscillator

Potential is $V = \frac{1}{2}m\omega^2 q^2$



$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \frac{p}{m} \\ -m\omega^2 q \end{pmatrix} .$$

Energy surfaces are ellipses,

$$E = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 .$$

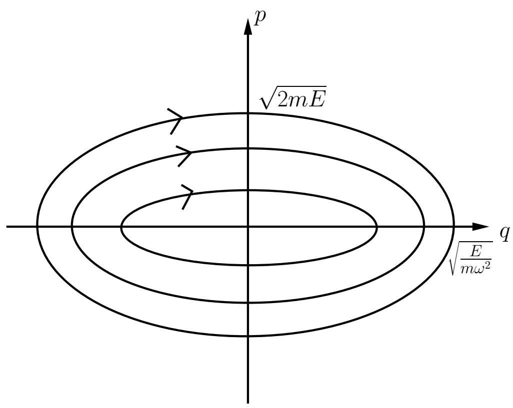
Motion:

$$q(t) = A \sin(\omega t + \delta) .$$

Fixed point at $(0, 0)$.

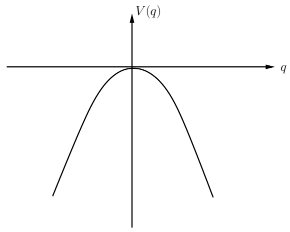
$$A = \begin{pmatrix} 0 & \frac{1}{m} \\ -m\omega^2 & 0 \end{pmatrix} \Longrightarrow \lambda = \pm i\omega ,$$

(Stable) elliptic fixed point at $(0, 0)$



3.5.2 Linear Repulsive Force (Repeller)

Potential is $V = -\frac{1}{2} m\gamma^2 q^2$



$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \frac{p}{m} \\ m\gamma^2 q \end{pmatrix} .$$

Energy surfaces are hyperbolae,

$$E = \frac{p^2}{2m} - \frac{1}{2} m\gamma^2 q^2 .$$

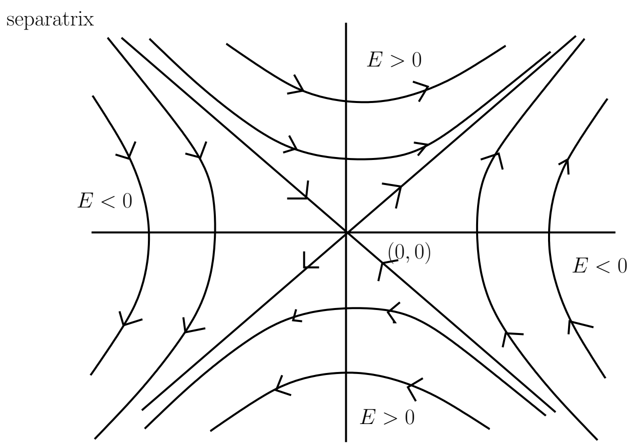
Motion:

$$q(t) = Ae^{\gamma t} + Be^{-\gamma t} .$$

Fixed point at $(0, 0)$.

$$A = \begin{pmatrix} 0 & \frac{1}{m} \\ m\gamma^2 & 0 \end{pmatrix} \Longrightarrow \lambda = \pm\gamma ,$$

(Unstable) hyperbolic fixed point at $(0, 0)$. As here $E_{separatrix} = 0 - 0 = 0$ (as goes through hyperbolic fixed point) then the separatrices are given by $p = \pm\frac{1}{2} m\gamma q$.



3.5.3 Free Rotations

$$\begin{pmatrix} \dot{\theta} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \frac{p}{I} \\ 0 \end{pmatrix} .$$

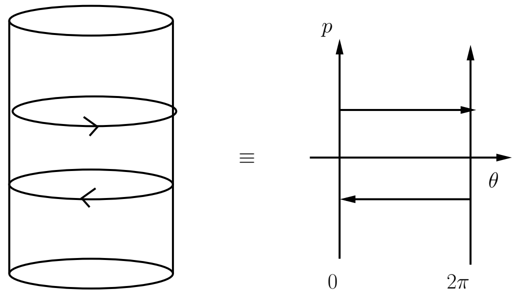
Energy surfaces are lines

$$E = \frac{p^2}{2I}\,,$$

where I is the moment of inertia. Motion:

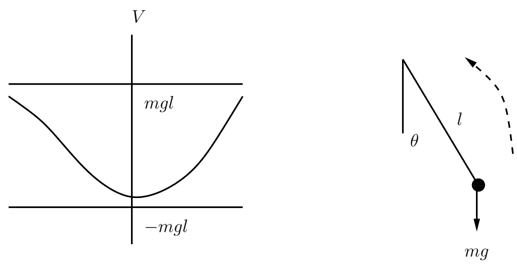
$$\theta = \frac{p_0}{I}t + \theta_0\,.$$

Line of fixed points at $p = 0$ (and $\lambda = 0$)



3.5.4 Vertical Pendulum

Potential is $V(\theta) = -mgl\cos\theta + [mgl]$



$$\begin{pmatrix} \dot\theta \\ \dot p \end{pmatrix} = \begin{pmatrix} \frac{p}{ml^2} \\ -mgl\sin\theta \end{pmatrix}.$$

Energy surfaces,

$$E = \frac{p^2}{2ml^2} - mgl\cos\theta\,,$$

Motion cannot be solved in terms of elementary functions.

(Stable) elliptic fixed point at $(0,0)$:

$$A|_{(0,0)} = \begin{pmatrix} 0 & \frac{1}{ml^2} \\ -mgl & 0 \end{pmatrix} \Longrightarrow \lambda = \pm i\sqrt{g/l}\,,$$