Hamiltonian Dynamics – Example sheet IV: 15.02.12

Semester 2 2012

1. A damped vertical pendulum has the equation of motion

$$\ddot{\theta} + 2\alpha\dot{\theta} + \omega^2 \sin\theta = 0\,,$$

where θ is the angle between the pendulum and the downward vertical. Determine the position and nature of the fixed points of the motion, and sketch the phase space trajectories for the cases $\alpha > \omega$ (over-damping), $\omega > \alpha > 0$ (under or light damping) and $\alpha = 0$ (no damping).

[Note: this is *not* a Hamiltonian system but a dissipative system.]

2. Describe the qualitative behaviour of the motion of a particle of unit mass in the cubic potential

$$V(q) = \tfrac{1}{2}\omega^2 q^2 - \tfrac{1}{3}\alpha q^3\,,$$

($\alpha > 0$). Determine the period of small oscillations in the neighbourhood of any stable fixed points of the motion.

Find the equation of the separatrix for the hyperbolic fixed point. If a particle moves from rest on the separatrix from the point $-q_0/2$, where $q_0 = \omega^2/\alpha$, show that at a later time

$$q(t) = q_0\left[1 - \tfrac{3}{2}\mathrm{sech}^2(\omega t/2)\right]\,.$$

3. A particle of mass m is constrained to slide under gravity on a smooth wire in the shape of a vertical circle with radius R, when the wire rotates about the vertical diameter with constant velocity Ω. Let ψ be the angular displacement of the particle from the downward vertical, with the centre of the circle as the origin. Show that the Hamiltonian of the system is

$$H = \frac{p^2}{2mR^2} - m\left(\frac{R^2\Omega^2}{2}\sin^2\psi + gR\cos\psi\right)\,,$$

where p is the momentum conjugate to ψ, and g is the acceleration due to gravity.

Determine the fixed points of the motion, discuss their stability and sketch the phase diagram. Determine the frequency of small oscillations about any elliptic fixed points.

4. Show that the origin is the only fixed point for the system

$$\begin{aligned} \dot{x} &= -y + \alpha x(\beta - x^2 - y^2)\,, \\ \dot{y} &= x + \alpha y(\beta - x^2 - y^2)\,, \end{aligned}$$

where α, β are real parameters with α fixed and positive while the control parameter β is allowed to take different values. Show that the character of the fixed point and the existence of a stable limit cycle depend on β.

[Hint: change immediately to polar co-ordinates.]

5. A particle of unit mass is moving in the *even* Hamiltonian

$$\begin{aligned} H(x,p) &= \tfrac{1}{2}p^2 + x(2-x) \qquad x > 0\,, \\ H(-x,p) &= H(x,p)\,. \end{aligned}$$

Find the fixed points of the Hamiltonian and classify them. Sketch (1) the potential, (2) the force $F(x) = -V'(x)$ and (3) the phase space showing fixed points, seperatrices and typical orbits taking particular care with the orbits crossing $x = 0$.

(a) For what range of energies is the motion (1) bounded libration, (2) unbounded?

(b) Prove that the period of librations of energy E is given by

$$T(E) = 2\sqrt{2}\cosh^{-1}(1/k(E)) \quad \text{where} \quad k(E) = \sqrt{1-E}\,.$$

(c) Given that $\cosh^{-1} y = \ln[y + \sqrt{y^2-1}]$, find the period of oscillations for small E. How does your result differ from that for small oscillations near a quadratic minimum of a potential?

(d) Show that as E approaches E_S (the energy on the separatrix) the period diverges logarithmically as

$$T(E) = -\sqrt{2}\ln(E_S - E)\,.$$

[June 1997]