Hamiltonian Dynamics – Example sheet I: 25.01.12

Semester 2 2012

1. A rod on a table rotates about one fixed end point with constant angular velocity ω. A bead of mass m slides without friction along the rod. Classify the constraints, find suitable generalised coordinates and solve Lagrange's equations. Discuss the result.

2. Show that Lagrange's equations of motion are unchanged when the following 'gauge' transformation is performed,

$$L \to L'(\underline{q}, \dot{\underline{q}}, t) = L(\underline{q}, \dot{\underline{q}}, t) + \frac{dF(\underline{q}, t)}{dt}\,.$$

3. A system of N particles described by the vector co-ordinates $\underline{r}_k$, $k = 1, 2, \ldots, N$ subject to $3N - f$ constraints can be expressed in terms of generalised co-ordinates q_i, $i = 1, 2, \ldots, f$ by

$$\underline{r}_k = \underline{r}_k(q_1, q_2, \ldots, q_f, t)\,.$$

Show that

$$\frac{\partial \dot{\underline{r}}_k}{\partial \dot{q}_i} = \frac{\partial \underline{r}_k}{\partial q_i}\,.$$

Show further that the kinetic energy of the system can be written

$$T = \sum_{k=1}^{N} \tfrac{1}{2} m_k \dot{\underline{r}}_k^2 = M_0 + \sum_{i=1}^{f} M_i \dot{q}_i + \tfrac{1}{2} \sum_{i=1}^{f} \sum_{j=1}^{f} M_{ij} \dot{q}_i \dot{q}_j\,,$$

and give expressions for M_0, M_i, M_{ij} in terms of the vector co-ordinates $\underline{r}_k$ and the time t.

Hence show that if the constraints are time-independent and the potential velocity independent, then the first two terms vanish and that also

$$\sum_{j=1}^{f} \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j = 2T\,.$$

4. Show that the motion of an electron of mass m and charge $-e$ moving with velocity $\dot{\underline{r}}$ in electromagnetic fields defined by a scalar potential ϕ and vector potential $\underline{A}$ can be described by a Lagrangian

$$L = \tfrac{1}{2} m \dot{\underline{r}}^2 + e\left(\phi - \frac{1}{c} \dot{\underline{r}} \cdot \underline{A}\right)\,.$$

The scalar and vector potentials are symmetrical about the z-axis and $A_\theta = f(r)/r$ using cylindrical polar co-ordinates (r, θ, z). When the electron is distant r_0 from the z-axis, its velocity is entirely in the (r, z) plane. Show that its angular velocity is about the z-axis is given by

$$\dot{\theta} = \frac{e}{mcr^2}\left[f(r) - f(r_0)\right] .$$

5. Revision of Lagrangian problems:

   Consider a system of N particles, labelled by $a = 1, 2, \cdots, N$, with constant masses m_a.

   Define what is meant by the centre of momentum frame.

   Show that the total kinetic energy T in an arbitrary inertial frame S may be written

   $$T = T_{\text{CM}} + \tfrac{1}{2}Mv^2$$

   where T_{CM} is the kinetic energy of the system with respect to the centre of momentum frame S_{CM}, M is the total mass of the system, and v is the magnitude of the velocity of the frame S_{CM} with respect to S.

   A rod of length L and mass M is constrained to move in a vertical plane. The upper end of the rod slides freely along a horizontal wire. Let x be the distance of the upper end of the rod from a fixed point, and let θ be the angle between the rod and the downward vertical.

   Show that the Lagrangian is

   $$L = \tfrac{1}{2}M\left(\dot{x}^2 + \tfrac{1}{3}L^2\dot{\theta}^2 + L\dot{x}\dot{\theta}\cos\theta\right) + \tfrac{1}{2}MgL\cos\theta .$$

   Hint: The moment of inertia of a thin rod of mass M and length L about its centre is $I = \frac{1}{12}ML^2$.

   Explain *briefly* why the Hamiltonian of this system is conserved, and why it is equal to the total energy.

   Find one further constant of the motion, and use the consequent conservation law to write down an expression for the total energy as a function of the variables θ and $\dot{\theta}$ only.

   If the rod is given an initial angular velocity ω when hanging vertically at rest, show that the maximum angle, θ_M, through which it can rise, satisfies

   $$\sin^2\left(\frac{\theta_M}{2}\right) = \frac{\omega^2 L}{24g} .$$