13 Symmetries in Particle Physics

Symmetries play in important role in particle physics. The mathematical description of symmetries uses group theory, examples of which are SU(2) and SU(3):

> A serious student of elementary particle physics should plan eventually to study this subject in far greater detail. (Griffiths P.115)

There is a relation between symmetries and conservation laws which is known as **Noether's theorem**. Examples of this in classical physics are:

- invariance under change of time → conservation of energy
- invariance under translation in space → conservation of momentum
- invariance under rotation → conservation of angular momentum

In particle physics there are many examples of symmetries and their associated conservation laws. There are also cases where a symmetry is broken, and the mechanism has to be understood. The breaking of electroweak symmetry and the associated Higgs field will be discussed in lecture 18.

13.1 Gauge Symmetries*

The Lagrangian is $L = T - V$, where T and V are the kinetic and potential energies of a system. It can be used to obtain the equations of motion. The Dirac equation follows from a Lagrangian of the form:

$$L = i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi \tag{13.1}$$

It can be seen that this Lagrangian is invariant under a phase transformation:

$$\psi \to e^{i\alpha}\psi \qquad \bar{\psi} \to e^{-i\alpha}\bar{\psi} \tag{13.2}$$

This is an example of a **gauge invariance**.

13.1.1 U(1) Symmetry of QED*

The Lagrangian for QED is written:

$$L = \bar{\psi}(i\gamma_\mu\partial^\mu - m)\psi + e\bar{\psi}\gamma_\mu A^\mu\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{13.3}$$

where A^μ represents the photon, and the second term can be thought of as $J_\mu A^\mu$ where $J_\mu = e\bar{\psi}\gamma^\mu\psi$ is an electromagnetic current. $F_{\mu\nu}$ is the electromagnetic field tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{13.4}$$

The gauge transformation is:

$$\psi \to e^{i\alpha}\psi \qquad A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha \tag{13.5}$$

This has the property that it conserves the current, and hence conserves electric charge. A mass term $mA_\mu A^\mu$ is forbidden in the Lagrangian by gauge invariance. This explains why the photon must be massless.

The gauge invariance of QED is described mathematically by a U(1) group.

13.1.2 SU(3) Symmetry of QCD*

SU(3) symmetry was introduced in Lecture 9 to describe QCD interactions. A rotation in colour space is written as:

$$U = e^{-i\alpha_a \cdot \lambda^a} \tag{13.6}$$

where α_a are the equivalent of "angles", and λ^a are the generators of SU(3) (equations (9.2) - (9.4)). QCD amplitudes can be shown to be invariant under this gauge transformation.

The transformations of the quark and gluon states are:

$$\mathrm{q} \to (1 + i\alpha_a\lambda^a)\mathrm{q} \qquad G^a_\mu \to G^a_\mu - \frac{1}{g_s}\partial_\mu\alpha_a - f_{abc}\alpha_b G^c_\mu \tag{13.7}$$

The Lagrangian for QCD is written:

$$L = \bar{\mathrm{q}}(i\gamma_\mu\partial^\mu - m)\mathrm{q} + g_s\bar{\mathrm{q}}\gamma_\mu\lambda^a G^\mu_a \mathrm{q} - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a \tag{13.8}$$

Where q represent the quark spinors, and compared to QED, the gluon states G^a_μ replace the photon, and g_s replaces e. The gluon field energy contains a term for the self-interactions of the gluons:

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - g_s f_{abc} G^b_\mu G^c_\nu \tag{13.9}$$

The absence of a mass term $mG^a_\mu G^\mu_a$ makes the gluon massless.

13.2 Flavour Symmetries

In Lecture 11 we met isospin symmetry, which is a flavour symmetry of the strong interactions between u and d quarks. It is described by SU(2), and can be extended to SU(3) with the addition of the s quark. The SU(3) symmetry is partially broken by the s quark mass. In principle this could be extended further to an SU(6) symmetry between all the quark flavours, but at this point the level of symmetry breaking becomes rather large. The interesting question, to which we do not yet have an answer, is what causes the breaking of quark flavour symmetry.

13.3 Discrete Symmetries

There are three important discrete symmetries: parity (P), charge conjugation (C) and time reversal (T). These are discussed in the following sections.

13.4 Parity

The parity operation P performs a spatial inversion through the origin:

$$P\psi(\vec{r}) = \psi(-\vec{r}) \tag{13.10}$$

This is NOT a mirror reflection through an axis, e.g. $\psi(x) \to \psi(-x)$. Many books get this wrong!

Applying parity twice restores the original state, $P^2 = 1$. From this the parity of a wavefunction $\psi(\vec{r})$ has to be either **even**, $P = +1$, or **odd**, $P = -1$. For example $\psi(x) = \cos kx$ is even, and $\psi(x) = \sin kx$ is odd.

The hydrogen atom wavefunctions are a product of a radial function $f(r)$ and the spherical harmonics $Y_L^m(\theta, \phi)$, where L and m are the orbital angular momentum of the state and its projection along an axis. In spherical polar coordinates the parity operation changes $\{r, \theta, \phi\} \longrightarrow \{r, \pi - \theta, \pi + \phi\}$. From the properties of Y_L^m the wavefunctions have parity $P = (-1)^L$. It is observed that single photon transitions between atomic states obey the selection rule $\Delta L = \pm 1$. From this it can be deduced that the **intrinsic parity** of the photon is:

$$(-1)^L = (-1)^{L\pm1} \times P_\gamma \qquad P_\gamma = -1 \tag{13.11}$$

The parity of the photon can also be obtained from the gauge symmetry of QED discussed in the previous section.

13.4.1 Intrinsic Parity of Fermions

Applying a spatial inversion to the Dirac equation gives

$$\left(i\gamma^0 \frac{\partial}{\partial t} - i\vec{\gamma} \cdot \vec{\nabla} - m \right) \psi(\vec{-r}, t) = 0 \tag{13.12}$$

This is not the same as the Dirac equation because there is a change of sign of the first derivative in the spatial coordinates. If we multiply from the left by γ^0 and use the relations $(\gamma^0)^2 = 1$ and $\gamma^0\gamma^i + \gamma^i\gamma^0 = 0$ ($i = 1, 2, 3$) we get back a valid Dirac equation:

$$\left(i\gamma^0 \frac{\partial}{\partial t} + i\vec{\gamma} \cdot \vec{\nabla} - m \right) \gamma^0\psi(\vec{-r}, t) = 0 \tag{13.13}$$

We identify the parity operator with γ^0:

$$\psi(\vec{r}, t) = P\,\psi(-\vec{r}, t) = \gamma^0\psi(-\vec{r}, t) \tag{13.14}$$

Applying this to the Dirac spinors (equation (5.24)):

$$P\,u_1 = +u_1 \qquad P\,u_2 = +u_2 \qquad P\,v_1 = -v_1 \qquad P\,v_2 = -v_2 \tag{13.15}$$

- The intrinsic parity of fermions is $P = +1$ (even)
- The intrinsic parity of antifermions is $P = -1$ (odd)

Parity is a multiplicative quantum number, so the parity of a many particle system is equal to the product of the intrinsic parities of the particles times the parity of the spatial wavefunction which is $(-1)^L$.

As an example, positronium is an e^+e^- atom with:

$$P(e^+e^-) = P_{e^-}P_{e^+}(-1)^L = (-1)^{L+1} \tag{13.16}$$

where L is the relative orbital angular momentum between the e^+ and e^-.

13.5 Charge Conjugation

Charge Conjugation is a discrete symmetry that reverses the sign of the electric charge, colour charge and magnetic moment of a particle. (It also reverses the values of the weak ispospin and hypercharge charges associated with the weak force, which we will meet in lecture 17.) Like the parity operator it satisfies $C^2 = 1$, and has possible eigenvalues $C = \pm 1$. Electromagnetism is C invariant, since Maxwell's equations apply equally to $+$ and $-$ charges. However the electromagnetic fields change sign under C, which means the photon has:

$$C_\gamma = -1 \tag{13.17}$$

For fermions charge conjugation changes a particle into an antiparticle, so fermions themselves are not eigenstates of C. Combinations of fermions can be eigenstates of C. for example, positronium has:

$$C(e^+e^-) = (-1)^{L+S} \tag{13.18}$$

where S is the sum of the spins which can be either 0 or 1.

Electromagnetic interactions are invariant under charge conjugation and parity, and conserve C and P quantum numbers.

We can also determine the P and C quantum numbers of mesons. The lowest pseudoscalar mesons (figure 11.2) have $J^{PC} = 0^{-+}$ and the vector mesons (figure 11.3) have $J^{PC} = 1^{--}$.

13.6 Time Reversal

Time reversal $T\psi(t) = \psi(-t)$, is another discrete symmetry operator with $T^2 = 1$, and possible eigenvalues $T = \pm 1$. The solutions of the Dirac equation describe antifermion states as equivalent to fermion states with the time and space coordinates reversed.

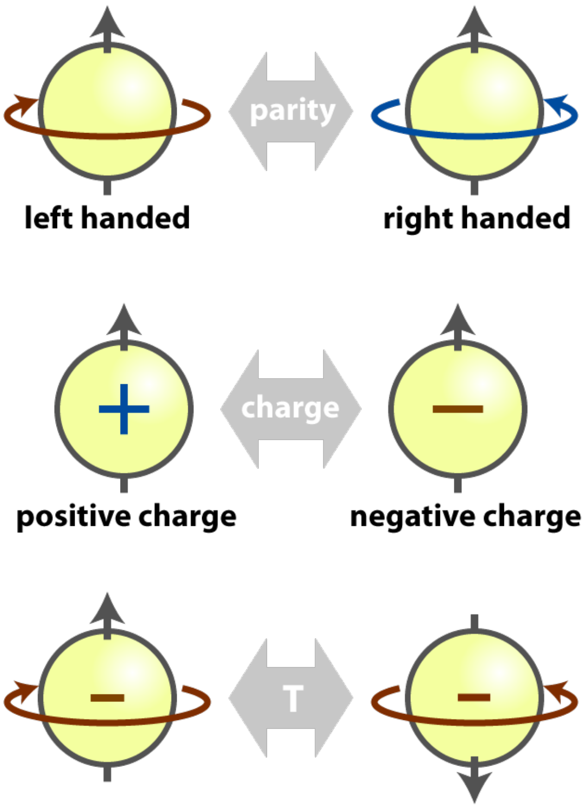

Figure 13.1: The action of the P (top), C (middle) and T (bottom) operators on a fundamental fermion. Note that the action of the operator twice on the state gives the original state back!

13.7 Summary of Discrete Symmetry Transformations

Figure 13.1 illustrates the actions of C, P and T on a fundamental fermion.

- A polar vector such as momentum, $\vec{p}$, transforms under parity $P\vec{p} = -\vec{p}$, $P = -1$.
- An axial vector such as angular momentum, $\vec{L} = \vec{r} \times \vec{p}$ transforms as $P\vec{L} = \vec{L}$, $P = +1$. The parity operator changes the direction of motion of a particle, but not the direction of the spin vector. As the helicity quantum number is a measure of the spin vector w.r.t. the momentum vector, parity changes a left-handed state into a right-handed state, and vice-versa.
- Charge conjugation reverses the charge, but does not change the direction of the spin vector or the momentum of a particle.
- Time reversal changes the sign of both the spin and momentum.

A summary of the effects of C, P and T on various quantities is shown in the table below.

Quantity	Notation	P	C	T
Position	$\vec{r}$	-1	$+1$	$+1$
Momentum (Vector)	$\vec{p}$	-1	$+1$	-1
Spin (Axial Vector)	$\vec{\sigma} = \vec{r} \times \vec{p}$	$+1$	$+1$	-1
Helicity	$\vec{\sigma}.\vec{p}$	-1	$+1$	$+1$
Electric Field	$\vec{E}$	-1	-1	$+1$
Magnetic Field	$\vec{B}$	$+1$	-1	-1
Magnetic Dipole Moment	$\vec{\sigma} \cdot \vec{B}$	$+1$	-1	$+1$
Electric Dipole Moment	$\vec{\sigma} \cdot \vec{E}$	-1	-1	-1
Transverse Polarization	$\vec{\sigma} \cdot (\vec{p_1} \times \vec{p_2})$	$+1$	$+1$	-1

13.8 Parity Violation in Weak Interactions

In contrast to electromagnetic and strong interacions it is found that weak interactions **maximally violate** both parity and charge conjugation symmetries.

The original evidence for parity violation came from the study of the β decay of polarized ^{60}Co, where it was observed that the electron was emitted preferentially in the direction opposite to the spin of the nucleus. The distribution of the decay electrons can be described by:

$$\frac{dN}{d\Omega} = 1 - \frac{\vec{\sigma} \cdot \vec{p}}{E} \tag{13.19}$$

The parity operator reverses the direction of the electron but not the spin of the nucleus, so the $\vec{\sigma} \cdot \vec{p}$ term is parity-violating.

Parity violation means that, for the weak interaction, there is a preferred spatial direction.

13.8.1 Parity Transformation of Vertex Terms

Recall that the vertex term for W-boson exchange is $g_W \gamma^\mu (1 - \gamma^5)/\sqrt{8}$. Under parity:

$$P(\gamma^\mu - \gamma^\mu \gamma^5) \rightarrow \gamma^\mu + \gamma^\mu \gamma^5 \tag{13.20}$$

the term changes.

Compare this to the QED and QCD vertex terms where the vertex terms stay the same and parity is conserved:

$$P(\gamma^\mu) \rightarrow \gamma^\mu \tag{13.21}$$

14 CP, CPT and CP Violation

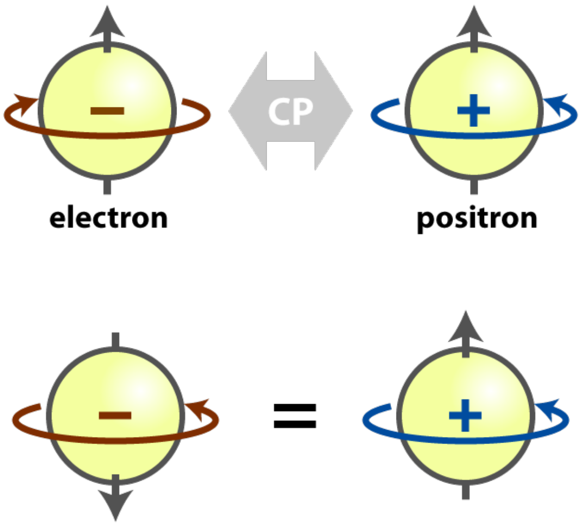

Figure 14.1: The action of the CP (top) and and CPT (bottom) operators on a fundamental fermion. The equals sign in the bottom plot represents CPT conservation.

14.1 CPT Theorem

The CPT theorem requires that all interactions that are described by localized Lorentz invariant gauge theories must be invariant under the combined operation of C, P and T in any order. The proof of the CPT theorem is based on very general field theoretic assumptions. It can be thought of as a statement about the invariance of Feynman diagrams under particle/antiparticle interchange, and interchange of the initial and final states.

The CPT theorem also means that the transformation properties of gauge theories under the discrete symmetries C, P and T are related to each other:

$$CP \leftrightarrow T \qquad CT \leftrightarrow P \qquad PT \leftrightarrow C \tag{14.1}$$

The first of these establishes that time reversal invariance is equivalent to CP invariance.

14.2 Tests of CPT Invariance

The CPT theorem predicts that particles and antiparticles must have the same mass and lifetime, but opposite electric charge and magnetic moment. Experimental tests of the CPT theorem have shown very precise agreement.

Some of the experimental evidence for CPT invariance are:

$$\begin{aligned} \frac{M(K^0) - M(\overline{K}^0)}{\frac{1}{2}[M(K^0) + M(\overline{\mathrm{K}}^0)]} &< 10^{-18} \\ \frac{\Gamma(K^0) - \Gamma(\overline{K}^0)}{\frac{1}{2}[M(K^0) + M(\overline{K}^0)]} &< 10^{-17} \\ Q(p) + Q(\bar{p}) &< 10^{-21}e \end{aligned} \tag{14.2}$$

14.3 CP Symmetry

Both charge conjugation and parity are found to be maximally violated in weak decays. However, experimental results suggest the combination CP is a nearly a conserved symmetry. CP turns a particle into its antiparticle with opposite helicity: it is a symmetry between matter and anti-matter CP is a conserved quantity in absolutely strong and electromagnetic interactions.

14.4 Sakharov Conditions to explain the Matter - Anti-Matter Asymmetry of the Universe*

In the Big Bang model of the universe there is an arrow of time, whereby time only goes forward, and therefore T may not be a valid symmetry of the universe. It is believed that matter and antimatter were originally created in equal amounts in the big bang. However we observe that we live in a matter dominated universe, with a baryon density compared to photons of $N_{\mathrm{b}}/N_{\gamma} = 10^{-9}$, and no evidence for primordial antibaryons. In 1966 Sakharov postulated three conditions that are necessary for our matter-dominated universe to exist:

- An epoch with no thermal equilibrium
- Baryon number violation
- CP Violation (or equivalently T violation)

Therefore CP violation is required to explain the difference between matter and anti-matter.

14.5 Neutrino States

The neutrino states are illustrated in figure 14.2.

A massless neutrino is purely left-handed, and a massless antineutrino is purely right-handed. The P operator reverses the helicity state. The C operator changes a neutrino into an antineutrino. Each of these operators by itself changes a physical state into

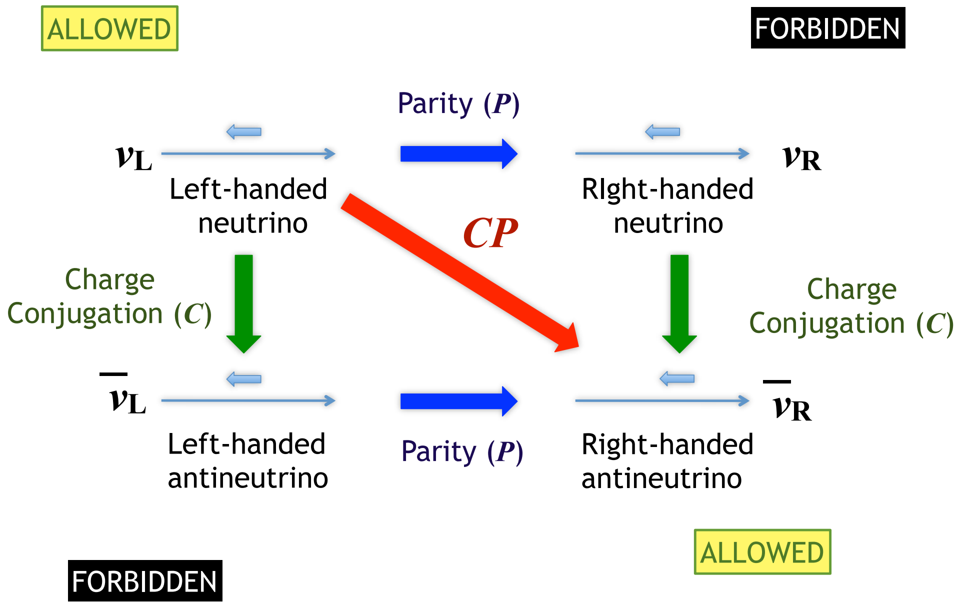

Figure 14.2: The operation of C and P on neutrinos.

a forbidden state, again showing that P and C must be maximally violated in weak interactions with neutrinos. The combined operator CP changes a left-handed neutrino into a right-handed antineutrino which is allowed.

This provides another motivation as to why to consider the symmetry of the combined CP operation.

14.6 Neutral Meson Mixing

Second order weak interactions can mix long-lived neutral mesons with their antiparticles. Mixing of the following mesons has been observed:

$$K^0(\bar{\mathrm{s}}\mathrm{d}), \quad D^0(\bar{\mathrm{c}}\mathrm{u}), \quad B^0(\bar{\mathrm{b}}\mathrm{d}), \quad B^0_{\mathrm{s}}(\bar{\mathrm{b}}\mathrm{s}) \tag{14.3}$$

The following transitions have been observed:

$$K^0 \leftrightarrow \overline{K}^0 \quad D^0 \leftrightarrow \overline{D}^0 \quad B^0 \leftrightarrow \overline{B}^0 \quad B^0_{\mathrm{s}} \leftrightarrow \overline{B}^0_{\mathrm{s}} \tag{14.4}$$

The Feynman digram describing the transition $K^0 \leftrightarrow \overline{K}^0$ is shown in figure 14.3. Note that this is a second order weak process as two W bosons are exchanged. The internal quarks can be any of the up-type quark, and need not be the same one on each side. The rate of the mixing is proportional to the CKM matrix elements of the quarks involved in the diagram. For example, the main contribution to neutral kaon mixing turns out to be from from internal charm quarks, therefore the rate of mixing is proportional to $V^*_{\mathrm{cd}}V_{\mathrm{cs}}$.

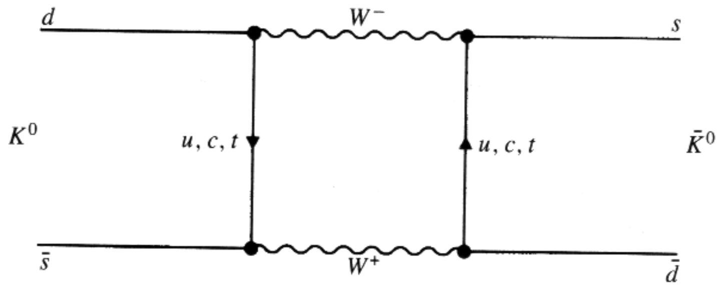

Figure 14.3: A Feynman diagram illustrating the second order weak processing which causes neutral meson mixing. Note that the rate of the mixing is proportional to the CKM matrix elements, e.g. $V_{\rm cd}^* V_{\rm cs}$.

14.6.1 Mixing of Neutral Kaons

A state that is initially K^0 or $\overline{K}^0$ will evolve as a function of time due to the mixing diagram:

$$\psi(t) = a(t)|K^0\rangle + b(t)|\overline{K}^0\rangle \qquad i\frac{d\psi}{dt} = \hat{H}\psi(t) \tag{14.5}$$

where $\hat{H}$ is the effective Hamiltonian which can be written in terms of 2×2 mass and decay matrices $\hat{M}$ and $\hat{\Gamma}$:

$$\hat{H} = \hat{M} - \frac{i}{2}\hat{\Gamma} = \begin{pmatrix} M_{11} - \frac{i}{2}\Gamma_{11} & M_{12} - \frac{i}{2}\Gamma_{12} \\ M_{21} - \frac{i}{2}\Gamma_{21} & M_{22} - \frac{i}{2}\Gamma_{22} \end{pmatrix} \tag{14.6}$$

The diagonal elements of these matrices are associated with flavour-conserving transitions, while the off-diagonal elements are associated with the mixing transitions $K^0 \leftrightarrow \overline{K}^0$. Therefore the off-diagonal elements (M_{12}, M_{21}) are proportional to the CKM matrix elements, such as $V_{\rm cd}^* V_{\rm cs}$.

The matrix $\hat{H}$ has two eigenvectors corresponding to the mass and weak decay eigenstates known as $K_{\rm L}$ and $K_{\rm S}$, K-long and K-short. The names are chosen as $K_{\rm L}$ is long-lived with a lifetime $\tau_L = 51$ ns and $K_{\rm S}$ has a much shorter lifetime of $\tau_S = 0.09$ ns.

The eigenstates can be expressed as a linear superposition of K^0 and $\overline{K}^0$:

$$|K_{\rm S}\rangle = p|K^0\rangle + q|\overline{K}^0\rangle \qquad |K_{\rm L}\rangle = p|K^0\rangle - q|\overline{K}^0\rangle \tag{14.7}$$

where $|q|^2 + |p|^2 = 1$ and:

$$\frac{q}{p} = \frac{M_{12}^* - \frac{i}{2}\Gamma_{12}^*}{M_{12} - \frac{i}{2}\Gamma_{12}} \tag{14.8}$$

The differences in the masses and decay widths of the weak eigenstates. These are measured to be:

$$\Delta m_K = m_{\rm L} - m_{\rm S} = (3.52 \pm 0.01) \times 10^{-12}\ \text{MeV} = 0.529 \times 10^{10}\ \text{s}^{-1} \tag{14.9}$$

$$\Delta\Gamma_K = \frac{1}{\tau_{\rm L}} - \frac{1}{\tau_{\rm S}} = 1.1 \times 10^{10}\ \text{s}^{-1} \tag{14.10}$$

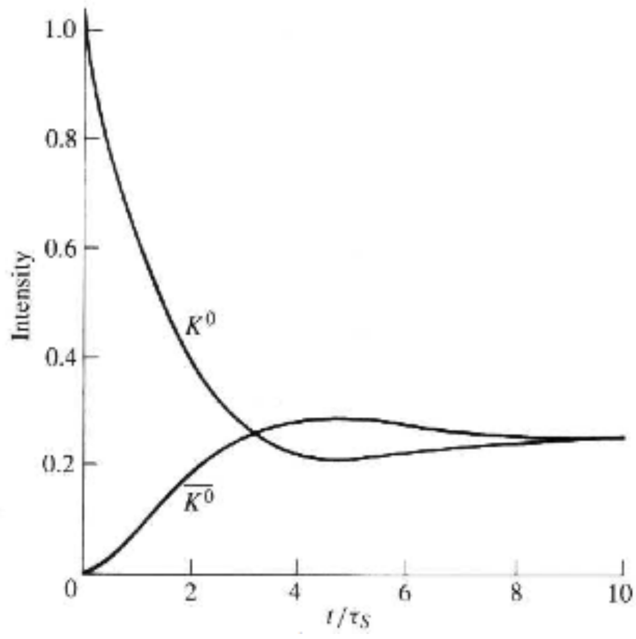

Figure 14.4: Time evolution of an initial K^0 state.

The mass difference Δm_K is very small compared to the neutral kaon mass!

Suppose we start with a initial beam that is purely K^0 (d$\bar{\text{s}}$). After a time t some of the beam will have mixed (evolved) into $\overline{K}^0$ (s$\bar{\text{d}}$), and some of the beam will have decayed: either as K_L or as K_S. The time evolution initial pure K^0 is given by:

$$|\psi_{K^0}(t)|^2 = \frac{1}{4}\left(e^{-\Gamma_\text{L}t} + e^{-\Gamma_\text{S}t} + 2e^{-\frac{(\Gamma_\text{L}+\Gamma_\text{S})}{2}t}\cos\Delta m_K t\right) \tag{14.11}$$

$$|\psi_{\overline{K}^0}(t)|^2 = \frac{1}{4}\left(e^{-\Gamma_\text{L}t} + e^{-\Gamma_\text{S}t} - 2e^{-\frac{(\Gamma_\text{L}+\Gamma_\text{S})}{2}t}\cos\Delta m_K t\right) \tag{14.12}$$

This time evolution is shown in figure 14.4. Neutral meson mixing leads to **flavour oscillations**, with a frequency given by the mass difference between the weak eigenstates.

15 Measuring CP Violation

15.1 CP Eigenstates of Kaons

Applying parity and charge conjugation to the K^0 and $\overline{K}^0$ states gives:

$$P\,|K^0\rangle = -|K^0\rangle \qquad P\,|\overline{K}^0\rangle = -|\overline{K}^0\rangle \tag{15.1}$$

$$CP\,|K^0\rangle = -|\overline{K}^0\rangle \quad CP\,|\overline{K}^0\rangle = -|K^0\rangle \tag{15.2}$$

K^0 and $\overline{K}^0$ are psudeoscalar mesons (see section 13.5), and therefore gain a minus sign under parity. Charge conjugation changes $\mathrm{d} \leftrightarrow \bar{\mathrm{d}}$ and $\mathrm{s} \leftrightarrow \bar{\mathrm{s}}$.

The CP eigenstates of neutral kaons are called K_1 and K_2:

$$K_1 = \frac{1}{\sqrt{2}}[K^0 + \overline{K}^0] \qquad K_2 = \frac{1}{\sqrt{2}}[K^0 - \overline{K}^0] \tag{15.3}$$

$$CP\,|K_1\rangle = +1\,|K_1\rangle \qquad CP\,|K_2\rangle = -1\,|K_2\rangle \tag{15.4}$$

If CP is violated the weak decay eigenstates are not the same as the CP eigenstates, we can write:

$$K_\mathrm{L} = \frac{1}{\sqrt{1+\epsilon^2}}[\epsilon K_1 + K_2] \qquad K_\mathrm{S} = \frac{1}{\sqrt{1+\epsilon^2}}[K_1 - \epsilon K_2] \tag{15.5}$$

where ϵ is a complex number.

As discussed above, the notation K_L and K_S refers to the long and short lifetimes:

$$\tau_\mathrm{L} = 5.2 \times 10^{-8}\ \mathrm{s} \qquad \tau_\mathrm{S} = 0.9 \times 10^{-10}\ \mathrm{s} \tag{15.6}$$

15.1.1 CP Violation in $K \to 2\pi$ Decays

The neutral kaon decay into both two and three pion final states, these states have:

$$CP|\pi\rangle = CP|\pi\rangle = +1 \tag{15.7}$$

$$CP|\pi^+\pi^-\rangle = CP|\pi^0\pi^0\rangle = +1 \tag{15.8}$$

$$CP|\pi^+\pi^-\pi^0\rangle = CP|\pi^0\pi^0\pi^0\rangle = -1 \tag{15.9}$$

If CP is conserved then the decays would be $K_1 \to 2\pi$ and $K_2 \to 3\pi$, with $K_1 \equiv K_\mathrm{S}$ and $K_2 \equiv K_\mathrm{L}$.

The decay $K_\mathrm{L} \to \pi^+\pi^-$ was first observed in 1964 was observed which violates CP. The ratios of decays are written as:

$$\eta_{+-} = \frac{K_\mathrm{L} \to \pi^+\pi^-}{K_\mathrm{S} \to \pi^+\pi^-} = \epsilon + \epsilon' \tag{15.10}$$

$$\eta_{00} = \frac{K_\mathrm{L} \to \pi^0\pi^0}{K_\mathrm{S} \to \pi^0\pi^0} = \epsilon - 2\epsilon' \tag{15.11}$$

where the parameter ϵ' represents a "direct" CP violation where the K_1 part of K_L decays directly to two pions.

The η have measured magnitudes and phases:

$$|\eta_{+-}| = 2.286(14) \times 10^{-3} \qquad \phi_{+-} = 43.4(7)^\circ \tag{15.12}$$

$$|\eta_{00}| = 2.276(14) \times 10^{-3} \qquad \phi_{00} = 43.6(8)^\circ \tag{15.13}$$

From a comparison of the charged and neutral pion decays:

$$\mathrm{Re}(\epsilon'/\epsilon) = 1.67(26) \times 10^{-3} \tag{15.14}$$

15.1.2 CP and T Violation in Semileptonic Decays

In semileptonic decays the charge of the lepton is given by the charge of the W boson. Thus a K^0 decay by an $\bar{\mathrm{s}} \to \bar{\mathrm{u}}$ transition gives an ℓ^+, decay by an $\mathrm{s} \to \mathrm{u}$ transition gives an ℓ^-. The charge of the lepton gives a **flavour tag** to the neutral kaon decay.

If there is no CP violation, the K_L is an equal superposition of K^0 and $\overline{K}^0$, so it should decay equally to ℓ^+ and ℓ^- with no charge asymmetry. If we add in the small amount of CP violation ϵ, then a charge asymmetry is predicted:

$$\delta_{SL} = \frac{\Gamma(\mathrm{K_L} \to \pi^-\ell^+\nu) - \Gamma(\mathrm{K_L} \to \pi^+\ell^-\bar{\nu})}{\Gamma(\mathrm{K_L} \to \pi^-\ell^+\nu) + \Gamma(\mathrm{K_L} \to \pi^+\ell^-\bar{\nu})} \tag{15.15}$$

$$\delta_{SL} = \frac{(1+\epsilon)^2 - (1-\epsilon)^2}{(1+\epsilon)^2 + (1-\epsilon)^2} = 2\mathrm{Re}(\epsilon) \tag{15.16}$$

The experimental measurement of this asymmetry is:

$$\delta_{SL} = 3.27(12) \times 10^{-3} \tag{15.17}$$

There is another elegant measurement that can be made with semileptonic decays that explicitly demonstrates time-reversal violation. We start with a pure K^0 or $\overline{K}^0$ state, and let it oscillate and then decay semileptonically. The T violation is observable as a rate asymmetry:

$$\Gamma(K^0 \to \overline{K}^0 \to \pi^+\ell^-\bar{\nu}) \neq \Gamma(\overline{K}^0 \to K^0 \to \pi^-\ell^+\nu) \tag{15.18}$$

The amount of T violation corresponds to the amount of CP violation, so CPT symmetry is preserved. A direct test of CPT violation in semileptonic decays would be:

$$\Gamma(\overline{K}^0 \to \pi^+\ell^-\bar{\nu}) \neq \Gamma(K^0 \to \pi^-\ell^+\nu) \tag{15.19}$$

The bounds on a CPT violating parameter δ_{CPT} in neutral kaon decays are actually only one order of magnitude below ϵ:

$$\delta_{CPT} = (2.9 \pm 2.7) \times 10^{-4} \tag{15.20}$$

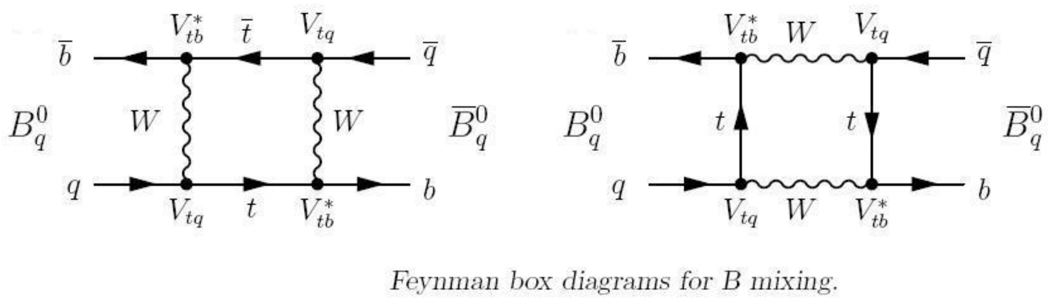

Figure 15.1: Feynman diagrams illustrating $B^0 \leftrightarrow \overline{B}^0$ and $B^0_\mathrm{s} \leftrightarrow \overline{B}^0_\mathrm{s}$ mixing.

15.1.3 Mixing of B mesons

A similar mixing occurs with neutral B-mesons: B^0 (also called B^0_d, $\mathrm{d\bar{b}}$) and B^0_s ($\mathrm{s\bar{d}}$), as illustrated in figure 15.1.

They are expected to mix in a similar way to the K^0 states, but in this case the mixing diagram is dominated by the top quark, and the off-diagonal elements of the mixing matrix (equation (14.6)) are given by:

$$M_{12} \propto (V_\mathrm{tb}V^*_\mathrm{td})^2 \qquad \frac{q}{p} = \frac{V^*_\mathrm{tb}V_\mathrm{td}}{V_\mathrm{tb}V^*_\mathrm{td}} \tag{15.21}$$

Oscillations of B^0 mesons have been observed with:

$$\Delta m_\mathrm{d} = 0.508(4)\ \mathrm{ps}^{-1} \qquad \tau_{B^0_\mathrm{d}} = 1.53(1)\ \mathrm{ps} \tag{15.22}$$

B_s oscillations were first observed at the Tevatron collider near Chicago in 2006 using an amplitude scan to Fourier analyse their B_s decays. They measured:

$$\Delta m_\mathrm{s} = 17.8(1)\ \mathrm{ps}^{-1} \qquad \tau_{B_\mathrm{s}} = 1.47(6)\ \mathrm{ps} \tag{15.23}$$

Note the much larger oscillation frequency which makes the direct observation of the oscillations difficult, although it should be possible at the LHC.

From the ratio of the two oscillation frequencies it is possible to determine:

$$\left|\frac{V_\mathrm{td}}{V_\mathrm{ts}}\right| = 0.206(1) \tag{15.24}$$

The main uncertainty in this ratio is now coming from the theoretical calculation of the hadronic properties of B mesons, the decay constants f_B and the "bag" constants B_B. It should be noted that most of the uncertainties cancel in the ratio, and the individual determinations of $|V_\mathrm{td}|$ and $|V_\mathrm{ts}|$ have theoretical errors which are $\times 10$ larger.

15.2 General Formalism for CP violation*

There is an excellent review of CP violation in the Particle Data Group compilation at http://pdg.lbl.gov/2011/reviews/rpp2011-rev-cp-violation.pdf.

In a more general notation the weak eigenstates are labelled M_{L} and M_H (for light and heavy mass), and are not assumed to be the same as the CP eigenstates:

$$M_{\mathrm{L}} = pM^0 + q\bar{M}^0 \qquad M_H = pM^0 - q\bar{M}^0 \tag{15.25}$$

with the normalisation $|p|^2 + |q|^2 = 1$.

We use the following notation for the mass and decay width differences:

$$\Delta m = M_H - M_{\mathrm{L}} \qquad \Delta\Gamma = \Gamma_H - \Gamma_{\mathrm{L}} \tag{15.26}$$

$$\Gamma = \frac{\Gamma_H + \Gamma_{\mathrm{L}}}{2} \qquad x = \frac{\Delta m}{\Gamma} \qquad y = \frac{\Delta\Gamma}{\Gamma} \tag{15.27}$$

The amplitudes for the decays of the flavour eigenstates M^0 and $\bar{M}^0$ to a final state f or $\bar{f}$, are written as A and $\bar{A}$. If the final state is a CP eigenstate $f = \bar{f}$, but A and $\bar{A}$ are not necessarily equal.

The time dependent decay rates of the flavour eigenstates to a CP eigenstate $M^0 \to f$ and $\bar{M}^0 \to f$, are given in the most general form by:

$$\frac{d\Gamma}{dt} = e^{-\Gamma t}\left[\alpha\cosh(\Delta\Gamma t) + \beta\cos(\Delta m t) + 2\mathrm{Re}[\gamma]\sinh(\Delta\Gamma t) - 2\mathrm{Im}[\gamma]\sin(\Delta m t)\right] \tag{15.28}$$

$$\alpha = |A|^2 + |\frac{q}{p}\bar{A}|^2 \quad \beta = |A|^2 - |\frac{q}{p}\bar{A}|^2 \quad \gamma = \frac{q}{p}A^*\bar{A} \tag{15.29}$$

$$\frac{d\bar{\Gamma}}{dt} = e^{-\Gamma t}\left[\bar{\alpha}\cosh(\Delta\Gamma t) - \bar{\beta}\cos(\Delta m t) + 2\mathrm{Re}[\bar{\gamma}]\sinh(\Delta\Gamma t) + 2\mathrm{Im}[\bar{\gamma}]\sin(\Delta m t)\right] \tag{15.30}$$

$$\bar{\alpha} = |\frac{p}{q}A|^2 + |\bar{A}|^2 \quad \bar{\beta} = |\frac{p}{q}A|^2 - |\bar{A}|^2 \quad \bar{\gamma} = \frac{p}{q}A\bar{A}^* \tag{15.31}$$

Note the changes in sign of the second and fourth terms in the decay rates.

The sin and cos terms give the mixing oscillations with frequency Δm. The amplitudes of these oscillations depend on γ, and include a possible CP violation through mixing.

15.3 Types of CP violation

There are three types of CP violation that can be observed:

- CP violation in the mixing amplitude, due to the mass eigenstates being different from the CP eigenstates, $|q/p| \neq 1$.
  In the neutral kaon system this is represented by the semileptonic charge asymmetry δ_{SL}.

- CP violation in the amplitudes A and $\bar{A}$ for decays to a particular final state, $|A/\bar{A}| \neq 1$ and phase differences between them. This is commonly known as **direct** CP violation. It does not require mixing, and can be found in both charged and neutral meson decays.
  In the decays $K_{L,S} \to 2\pi$ it is represented by ϵ'.

- CP violation in the interference between mixing and decay amplitudes, which requires an overall weak phase $\mathrm{Im}[\lambda] \neq 0$, where $\lambda = q\bar{A}/pA$.
  In the decays $K_{L,S} \to 2\pi$ it is represented by ϵ.

15.4 The CKM Matrix Revisited

Recall that the CKM matrix describes the difference between the mass and decay eigenstates of the down-type quarks:

$$\begin{pmatrix} \mathrm{d}' \\ \mathrm{s}' \\ \mathrm{b}' \end{pmatrix} = \begin{pmatrix} V_{\mathrm{ud}} & V_{\mathrm{us}} & V_{\mathrm{ub}} \\ V_{\mathrm{cd}} & V_{\mathrm{cs}} & V_{\mathrm{cb}} \\ V_{\mathrm{td}} & V_{\mathrm{ts}} & V_{\mathrm{tb}} \end{pmatrix} \begin{pmatrix} \mathrm{d} \\ \mathrm{s} \\ \mathrm{b} \end{pmatrix} \tag{15.32}$$

To ensure that the number of quarks is conserved, the CKM matrix must be unitary, that is $V_{\mathrm{CKM}}^{\dagger} V_{\mathrm{CKM}} = 1$, or:

$$\begin{pmatrix} V_{\mathrm{ud}}^* & V_{\mathrm{cd}}^* & V_{\mathrm{td}}^* \\ V_{\mathrm{us}}^* & V_{\mathrm{cs}}^* & V_{\mathrm{ts}}^* \\ V_{\mathrm{ub}}^* & V_{\mathrm{cb}}^* & V_{\mathrm{tb}}^* \end{pmatrix} \begin{pmatrix} V_{\mathrm{ud}} & V_{\mathrm{us}} & V_{\mathrm{ub}} \\ V_{\mathrm{cd}} & V_{\mathrm{cs}} & V_{\mathrm{cb}} \\ V_{\mathrm{td}} & V_{\mathrm{ts}} & V_{\mathrm{tb}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{15.33}$$

This implies nine **unitarity relations**, e.g. the most important one is:

$$V_{\mathrm{ud}} V_{\mathrm{ub}}^* + V_{\mathrm{td}} V_{\mathrm{tb}}^* + V_{\mathrm{cd}} V_{\mathrm{cb}}^* = 0 \tag{15.34}$$

As the elements of the matrix are simply complex numbers, equation (15.34) represents a triangle in the complex plane, as shown in figure 15.2. The angles and the sides of the triangle represent different combinations of CKM matrix elements. Notes that the triangle will only have a non-zero area if there is a relative complex phase between each of the sides.

Looking at the right-hand side of figure 15.2, the lengths of the sides are:

$$\left| \frac{V_{\mathrm{ud}} V_{\mathrm{ub}}^*}{V_{\mathrm{cd}} V_{\mathrm{cb}}^*} \right| \qquad \left| \frac{V_{\mathrm{td}} V_{\mathrm{tb}}^*}{V_{\mathrm{cd}} V_{\mathrm{cb}}^*} \right| \tag{15.35}$$

and the angles are:

$$\alpha \equiv \arg\left(-\frac{V_{\mathrm{td}} V_{\mathrm{tb}}^*}{V_{\mathrm{ud}} V_{\mathrm{ub}}^*} \right) \qquad \beta \equiv \arg\left(-\frac{V_{\mathrm{cd}} V_{\mathrm{cb}}^*}{V_{\mathrm{td}} V_{\mathrm{tb}}^*} \right) \qquad \gamma \equiv \arg\left(-\frac{V_{\mathrm{ud}} V_{\mathrm{ub}}^*}{V_{\mathrm{cd}} V_{\mathrm{cb}}^*} \right) \tag{15.36}$$

This unitarity triangle is often use to present measurements of CP violation in B-meson decay. A summary of the measured constraints on the lengths is shown in figure 15.3.

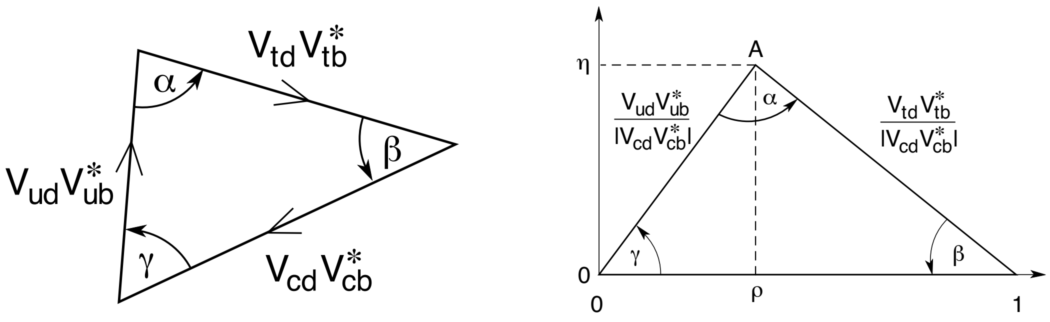

Figure 15.2: Illustration of the unitarity triangle. Left: equation (15.34); right: Dividing through by $V_{\rm cd} V_{\rm cb}^*$.

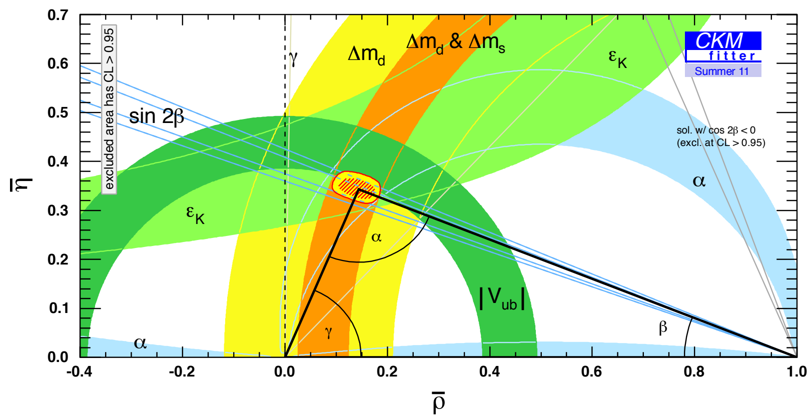

Figure 15.3: Fit of all relevant experimental data to the unitarity triangle paramters.

15.5 The Wolfenstein Parameterisation of the CKM Matrix

The Wolfenstein Parameterisation is an expansion of the CKM matrix in powers of a parameter $\lambda = V_{\text{us}} \approx 0.22$:

$$V_{\text{CKM}} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4) \tag{15.37}$$

The parameterisation reflects almost diagonal nature of CKM matrix:

- The diagonal elements $V_{\text{ud}}, V_{\text{cs}}, V_{\text{tb}}$ are close to 1.
- The elements $V_{\text{us}}, V_{\text{cd}} \sim \lambda$ are equal.
- The elements $V_{\text{cb}}, V_{\text{ts}} \sim \lambda^2$ are equal.
- The elements $V_{\text{ub}}, V_{\text{td}} \sim \lambda^3$ are very small.

The diagonal structure means down quark mass eigenstate is almost equal to down quark weak eigenstate similarly for strange and bottom mass eigenstates Note that the complex phase η only appears in the very small elements, and is thus hard to measure.

15.6 CP Violation in B Meson Decays

All three types are CP violation are expected to occur in the decays of neutral B mesons. Due to the dominance of the t quark contribution inside mixing and penguin diagrams, and the presence of the suppressed CKM couplings V_{ub} and V_{td}, measurements of CP violation in B decays provide important additional information compared to the neutral kaon system.

15.6.1 $B^0 \to J/\psi \text{K}_\text{S}$ and $\sin 2\beta$*

CP violation through interference between mixing and decay amplitudes was first observed in the decay $B^0 \to J/\psi \text{K}_\text{S}$ in 2001 by the BaBar and Belle experiments. For this decay:

$$\lambda = \frac{q\bar{A}_f}{pA_f} = \left(\frac{V_{\text{tb}}^* V_{\text{td}}}{V_{\text{tb}} V_{\text{td}}^*}\right)\left(\frac{V_{\text{cs}}^* V_{\text{cb}}}{V_{\text{cs}} V_{\text{cb}}^*}\right)\left(\frac{V_{\text{cs}}^* V_{\text{cd}}}{V_{\text{cs}} V_{\text{cd}}^*}\right) \tag{15.38}$$

where the three set of CKM factors respectively account for B^0 mixing, the $B \to J/\psi \text{K}_\text{S}$ decay amplitude and final state K^0 mixing.

15.6.2 $B^0 \to \pi\pi$, $B^0 \to \rho\rho$ and the angle α*

A similar measurement of time-dependent CP asymmetries can be made with the rare hadronic final states $B^0 \to \pi^+\pi^-$ and $B^0 \to \rho^+\rho^-$. In this case the decay amplitude is proportional to V_{ub}, and the angle α is measured which is the complex phase between V_{td} and V_{ub} in the Standard Model.

The BaBar experiment obtains $\alpha = (96 \pm 6)^\circ$ from a fit to $B \to \rho\rho$ decays where the penguin diagram has a much smaller effect.

15.6.3 Direct CP violation in $B \to K\pi$*

In 2004 the BaBar and Belle experiments made the first observation of a direct CP violation in the decay amplitudes for $B^0 \to K\pi$ decays:

$$A_{CP} = \frac{\Gamma(\bar{B}^0 \to K^-\pi^+) - \Gamma(B^0 \to K^+\pi^-)}{\Gamma(\bar{B}^0 \to K^-\pi^+) - \Gamma(B^0 \to K^+\pi^-)} \tag{15.39}$$

The latest world average for this is $A_{CP} = -0.10 \pm 0.01$.

15.7 CP Violation in D mesons

CP violation in D-mesons (mesons containing one charm quark) was observed for the first time in 2011 by the LHCb collaboration at CERN.

The collaboration measured the CP asymmetry between decays of D^0 and $\overline{D}^0$ mesons to kaons and pions:

$$\mathcal{A}_{CP} = \frac{\Gamma(D^0 \to K^+K^-) - \Gamma(\overline{D}^0 \to K^+K^-)}{\Gamma(D^0 \to K^+K^-) + \Gamma(\overline{D}^0 \to K^+K^-)} - \frac{\Gamma(D^0 \to \pi^+\pi^-) - \Gamma(\overline{D}^0 \to \pi^+\pi^-)}{\Gamma(D^0 \to \pi^+\pi^-) + \Gamma(\overline{D}^0 \to \pi^+\pi^-)} \tag{15.40}$$

The measured asymmetry was: $\mathcal{A}_{CP} = [-0.82 \pm 0.21(\mathrm{stat}) \pm 0.11(\mathrm{syst})]\%$, a clear difference between the behaviour of D^0 and $\overline{D}^0$ mesons is observed!