Particle Physics

Dr Victoria Martin, Spring Semester 2013

Lecture 4: Dirac Spinors

- ★ Schrödinger Equation
- ★ Klein-Gordon Equation
- ★ Dirac Equation
- ★ Spinors
- ★ Spin, helicity and chirality

Schrödinger Equation

- Classical energy-momentum relationship:

$$E = \frac{p^2}{2m} + V$$

- Substitute QM operators:

$$\hat{p} = -i\hbar\vec{\nabla} \qquad \hat{E} = i\hbar\frac{\partial}{\partial t}$$

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\hbar^2\frac{\nabla^2}{2m} + V\right)\psi = \hat{H}\psi$$

Schrödinger equation!

- 1st order in $\partial/\partial t$; 2nd order in $\partial/\partial x$. Space and time not treated equally.

Klein-Gordon Equation

- Relativistic energy-momentum relationship is:

$$E^2 = \vec{p}^{\,2} c^2 + m^2 c^4$$

- Substitute the operators:

$$\hat{p} = -i\hbar\vec{\nabla} \qquad \hat{E} = i\hbar\frac{\partial}{\partial t}$$

- To give the **Klein-Gordon** equation:

$$-\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} + \nabla^2\psi = \left(\frac{mc}{\hbar}\right)^2 \psi$$

- **The Klein-Gordon equation describes spin-0 bosons.** Solutions are plane waves (see lecture 3):

$$\psi = e^{-ip\cdot x} \qquad p\cdot x = p^\mu x_\mu = \hbar(\vec{k}\cdot\vec{x} - \omega t)$$

- KG equation is 2nd order in $\partial/\partial t$ and $\partial/\partial x$

Negative Energy & the Dirac Equation

- The relativistic energy-momentum equation is quadratic, negative energy solutions are possible:

$$E^2 = \vec{p}^{\,2} + m^2 \qquad \Rightarrow \qquad E = \pm\sqrt{\vec{p}^{\,2} + m^2}$$

- Dirac searched for 1st order relationship between energy and momentum, using coefficients $\boldsymbol{\alpha}^1$ $\boldsymbol{\alpha}^2$ $\boldsymbol{\alpha}^3$ and $\boldsymbol{\beta}$

$$\hat{E}\,\psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\psi = i\frac{\partial\psi}{\partial t}$$

- Need to find solutions for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$

Dirac Equation

- Solution is more elegant defining $\gamma^0 \equiv \beta, \gamma^1 \equiv \beta\alpha^1, \gamma^2 \equiv \beta\alpha^2, \gamma^3 \equiv \beta\alpha^3$
- The Dirac equation can be written (with $c = \hbar = 1$) as:

$$i\left(\gamma^0 \frac{\partial\psi}{\partial t} + \vec{\gamma}\cdot\vec{\nabla}\right)\psi = m\psi$$

in covariant notation: $i\gamma^\mu \partial_\mu \psi = m\psi$

- Multiplying the Dirac equation by its complex conjugate must give KG:

$$\left(-i\gamma^0 \frac{\partial}{\partial t} - i\vec{\gamma}\cdot\vec{\nabla} - m\right)\left(i\gamma^0 \frac{\partial}{\partial t} + i\vec{\gamma}\cdot\vec{\nabla} - m\right) = 0$$

- This leads to a set of conditions on the four coefficients γ^μ:

$$(\gamma^0)^2 = 1 \quad (\gamma^1)^2 = -1 \quad (\gamma^2)^2 = -1 \quad (\gamma^3)^2 = -1$$

$$\{\gamma^i, \gamma^j\} = \gamma^i\gamma^j + \gamma^j\gamma^i = 0$$

γ^μ are unitary and anticommute

The Gamma Matrices - 1

- To satisfy unitarity and anti-commutation the γ^μ must be at least 4 × 4 matrices.
- More than one representation. The usual one is:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- γ^μ are not tensors or four vectors! They do remain constant under Lorentz transformations

The Gamma Matrices - 2

- Gamma Matrices are also often written in a 2x2 form:

$$\gamma^0 = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}$$

- where $\mathbf{I}$ and $\mathbf{0}$ are the 2×2 identity and null matrices:

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

- and the $\boldsymbol{\sigma}^i$ are the 2×2 Pauli spin matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dirac Equation and Solution

- In matrix notation:

$$\begin{pmatrix} i\frac{\partial}{\partial t} - m & 0 & i\frac{\partial}{\partial z} & i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \\ 0 & i\frac{\partial}{\partial t} - m & i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} & -i\frac{\partial}{\partial z} \\ -i\frac{\partial}{\partial z} & -i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} & -i\frac{\partial}{\partial t} - m & 0 \\ -i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} & i\frac{\partial}{\partial z} & 0 & -i\frac{\partial}{\partial t} - m \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- In co-variant notation:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

- Solutions ψ to the Dirac Equation have a:
  - phase term: $e^{-ip\cdot x}$
  - Dirac spinor term, a function of the four-momentum: $u(p^\mu)$

$$\psi = u(p^\mu)e^{-ip\cdot x}$$ with u solution to $$(\gamma^\mu p_\mu - m)u = 0$$

Solutions to the Dirac Equation

- Dirac equation: $$(i\gamma^\mu \partial_\mu - m)\,\psi = 0$$

$$\left(i\gamma^0 \frac{\partial}{\partial t} - i\gamma^1 \frac{\partial}{\partial x} - i\gamma^2 \frac{\partial}{\partial y} - i\gamma^3 \frac{\partial}{\partial z} - m\right)\psi = 0$$

- Solve for a particle at rest, $p^\mu = (m, \mathbf{0})$, to illustrate main features of the solutions

$$\psi = u(p^\mu)e^{-ip\cdot x} = u(p^\mu)e^{-imt}$$

- Dirac equation becomes: $$(\gamma^0 E - m)\,u(p^\mu) = 0$$

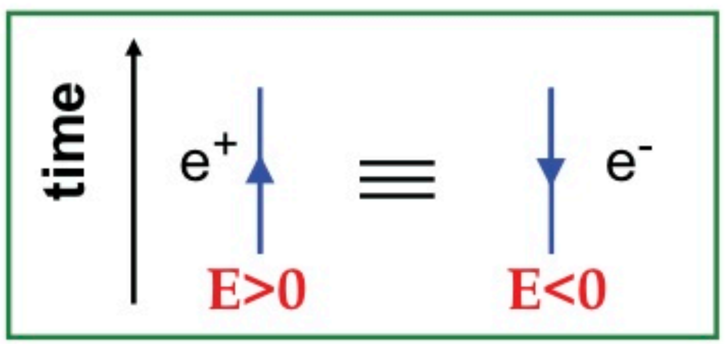
$$\begin{pmatrix} E-m & 0 & 0 & 0 \\ 0 & E-m & 0 & 0 \\ 0 & 0 & -E-m & 0 \\ 0 & 0 & 0 & -E-m \end{pmatrix} \begin{pmatrix} u^1(p^\mu) \\ u^2(p^\mu) \\ u^3(p^\mu) \\ u^4(p^\mu) \end{pmatrix} = 0$$

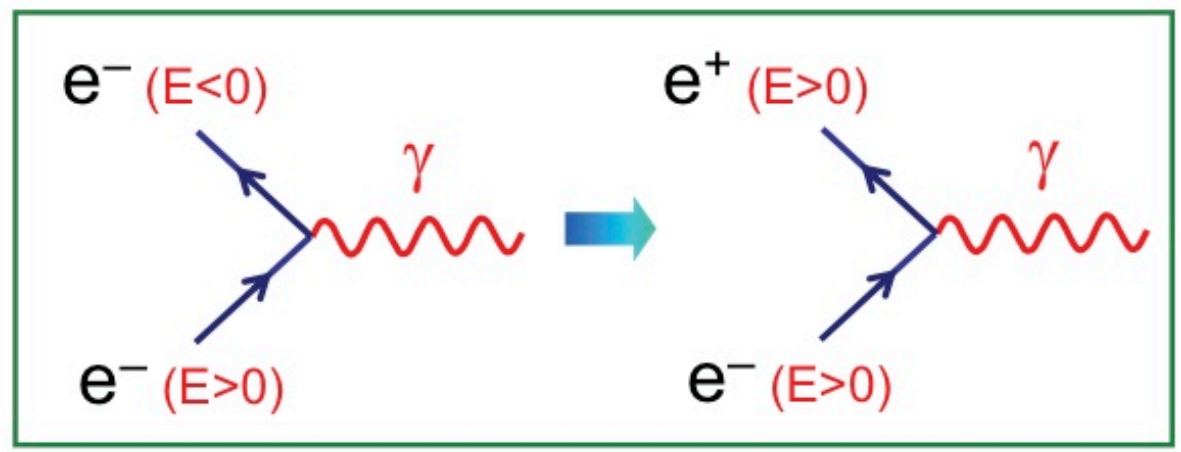
- Four energy eigenstates:
  - u^1 and u^2 with $E = +m$
  - u^3 and u^4 with $E = -m$

Negative Energy Solutions

- We can't escape negative energy solutions. How should we interpret them?
- **Modern Feynman-Stückelberg Interpretation:**

  A negative energy solution is a negative energy particle which propagates backwards in time or equivalently a positive energy anti-particle which propagates forwards in time.





$$e^{-i(-E)(-t)} \to e^{-iEt}$$

- This is why in Feynman diagrams the backwards pointing lines represent anti-particles.

Discovery of Positron

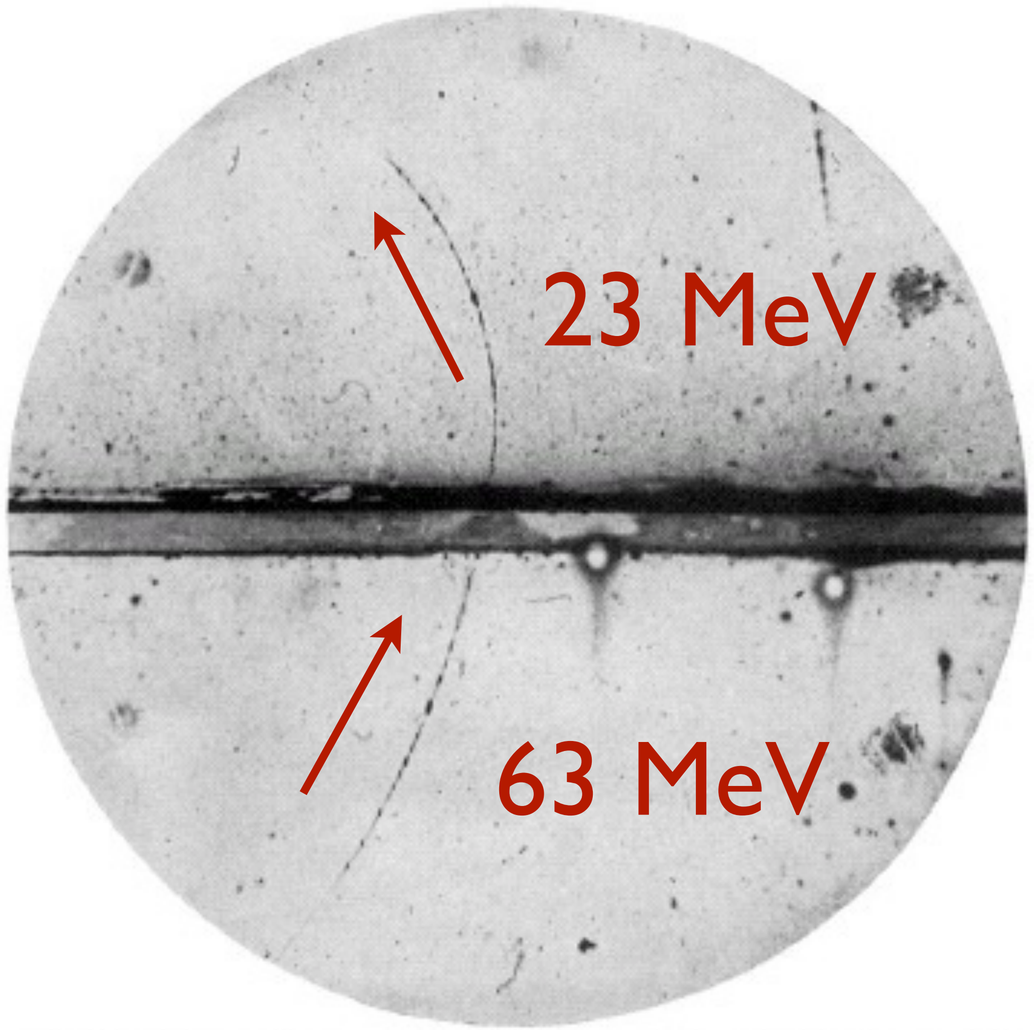
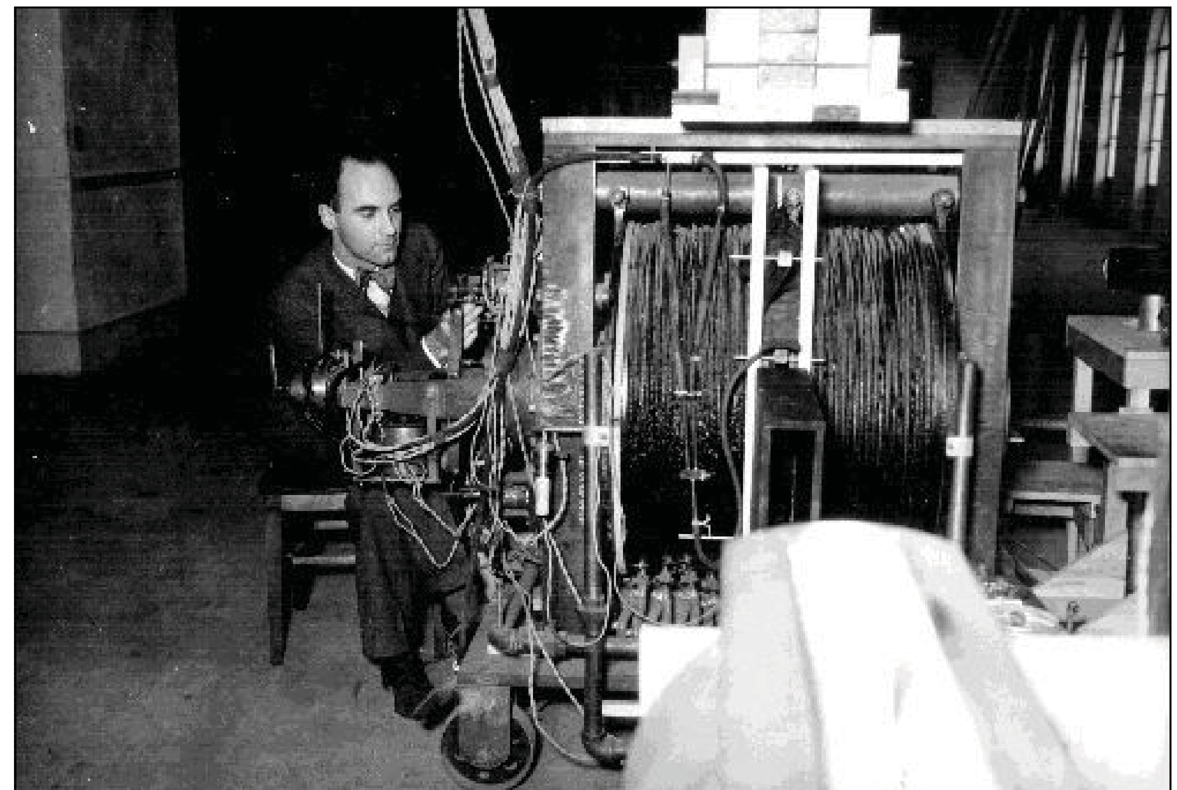


Fig. 1. A 63 million volt positron ($H\rho = 2.1\times10^5$ gauss-cm) passing through a 6 mm lead plate and emerging as a 23 million volt positron ($H\rho = 7.5\times10^4$ gauss-cm). The length of this latter path is at least ten times greater than the possible length of a proton path of this curvature.

C.D.Anderson, Phys Rev 43 (1933) 491



- e^+ enters at bottom, slows down in the lead plate - know direction
- Curvature in **B-field** shows that it is a **positive particle**
- Can't be a proton as would have stopped in the lead

Solutions

- Making the equation first order in all derivatives introduces new degrees of freedom!
- The four solutions represent the **four** possible states of a fermion.
- The u are 1 x 4 matrices - **spinors** or **Dirac spinors** (not four-vectors)!

- Using the electron as an example:
  - u^1 represents an electron ($E = m$) with spin-up
  - u^2 represents an electron ($E = m$) with spin-down
  - u^3 represents a positron ($E = -m$) with spin-down
  - u^4 represents a positron ($E = -m$) with spin-up

- $u^3(p)$ and $u^4(p)$ are often written as $v^1(p^\mu)=u^4(-p^\mu)$ and $v^2(p^\mu)=u^3(-p^\mu)$

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix} = \begin{pmatrix} u^1(p^\mu) \\ u^2(p^\mu) \\ u^3(p^\mu) \\ u^4(p^\mu) \end{pmatrix} e^{-ip\cdot x} = \begin{pmatrix} u^1(p^\mu) \\ u^2(p^\mu) \\ v^2(-p^\mu) \\ v^1(-p^\mu) \end{pmatrix} e^{-ip\cdot x}$$

Spinors moving and at rest

- For a particle at rest spinors take the trivial form:

$$u^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad u^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad u^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad u^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- For the moving particles (derivation see Griffiths Pp. 231-234):

Fermions:

$$u^1 = \begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ (p_x+ip_y)/(E+m) \end{pmatrix} \quad u^2 = \begin{pmatrix} 0 \\ 1 \\ (p_x-ip_y)/(E+m) \\ -p_z/(E+m) \end{pmatrix}$$

Antifermions:

$$v^2 = \begin{pmatrix} p_z/(E+m) \\ (p_x+ip_y)/(E+m) \\ 1 \\ 0 \end{pmatrix} \quad v^1 = \begin{pmatrix} (p_x-ip_y)/(E+m) \\ -p_z/(E+m) \\ 0 \\ 1 \end{pmatrix}$$

Where we have changed notation for antiparticles from $u^3(p) \rightarrow v^2(-p)$ and $u^4(p) \rightarrow v^1(-p)$

Helicity

- Spin is usually defined w.r.t the z-axis → not Lorentz invariant.
- Define helicity, $\hat{h}$, the component of the spin along direction of flight.

$$\hat{h} = \frac{\vec{S} \cdot \vec{p}}{|\vec{S}||\vec{p}|} = \frac{2\vec{S} \cdot \vec{p}}{|\vec{p}|}$$

- For a $S=½$ fermion, the project of spin along any axis can only be ±½.
- For a $S=½$ fermion, eigenvalues of $\hat{h}$ are ± 1.
- We call $h=+1$, “right-handed”, $h=-1$ “left handed”.

- Massless fermions with ($p=E$) are purely left-handed (only u^2)
- Massless antifermions are purely right-handed (only v^1)
- For massive particles helicity is still not Lorentz invariant: we can boost to frame such that particle direction of flight reverses

Chirality and Handedness

- Chirality is a Lorentz invariant quantify: identical to helicity for massless particles.
- $S=\frac{1}{2}$ fermions have two chiral states: left-handed and right-handed.

- Defined using chiral **projection operators** P_L and P_R:
- **LH projection** operator $P_L = (1 - \gamma^5)/2$ projects out **left-handed chiral** state
- **RH projection** operator $P_R = (1 + \gamma^5)/2$ projects out **right-handed chiral** state

where $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ is 4×4 matrix:

$$\gamma^5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Chiral Projection Operators and γ^5

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\gamma^5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- Properties:
  - ➡ Unitary $(\gamma^5)^2 = 1$
  - ➡ Anti commutes with all other γ matrices: $\{\gamma^5, \gamma^i\} = \gamma^5\gamma^i + \gamma^i\gamma^5 = 0$.

- Left and right handed component of a fermion state are $\psi_L = P_L\,\psi$, $\psi_R = P_R\,\psi$

- $P_L + P_R = 1 \Rightarrow \psi = P_L\,\psi + P_R\,\psi$
  - A state can always be written as the sum of LH and RH components

- $P_L^2 = P_L \qquad P_R^2 = P_R \qquad P_L\,P_R = 0$
  - No overlap between the LH and RH components

Summary and Reading List

- The Dirac Equation describes spin-½ particles.

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

- Solutions include four component **spinors**, u and v.

$$(\gamma^\mu p_\mu - m)u = 0 \qquad (\gamma^\mu p_\mu + m)v = 0$$

$$\psi = u(p)e^{-ip\cdot x} \qquad \psi = v(p)e^{-ip\cdot x}$$

- With γ^μ, $\mu=0,1,2,3$ the 4 × 4 Gamma matrices
- The four solutions describe the different states of the electron *e.g.* left-handed electrons, right-handed electrons, right-handed positrons, left-handed positrons

- We use chiral projection operators to define left-handed and right-handed states
- Any particle can be written in terms of left handed and right handed components: $\psi = (1-\gamma^5)\psi + (1+\gamma^5)\psi = \psi_L + \psi_R$

- Next Lecture: The Electromagnetic Force. Griffiths 7.5 & 7.6