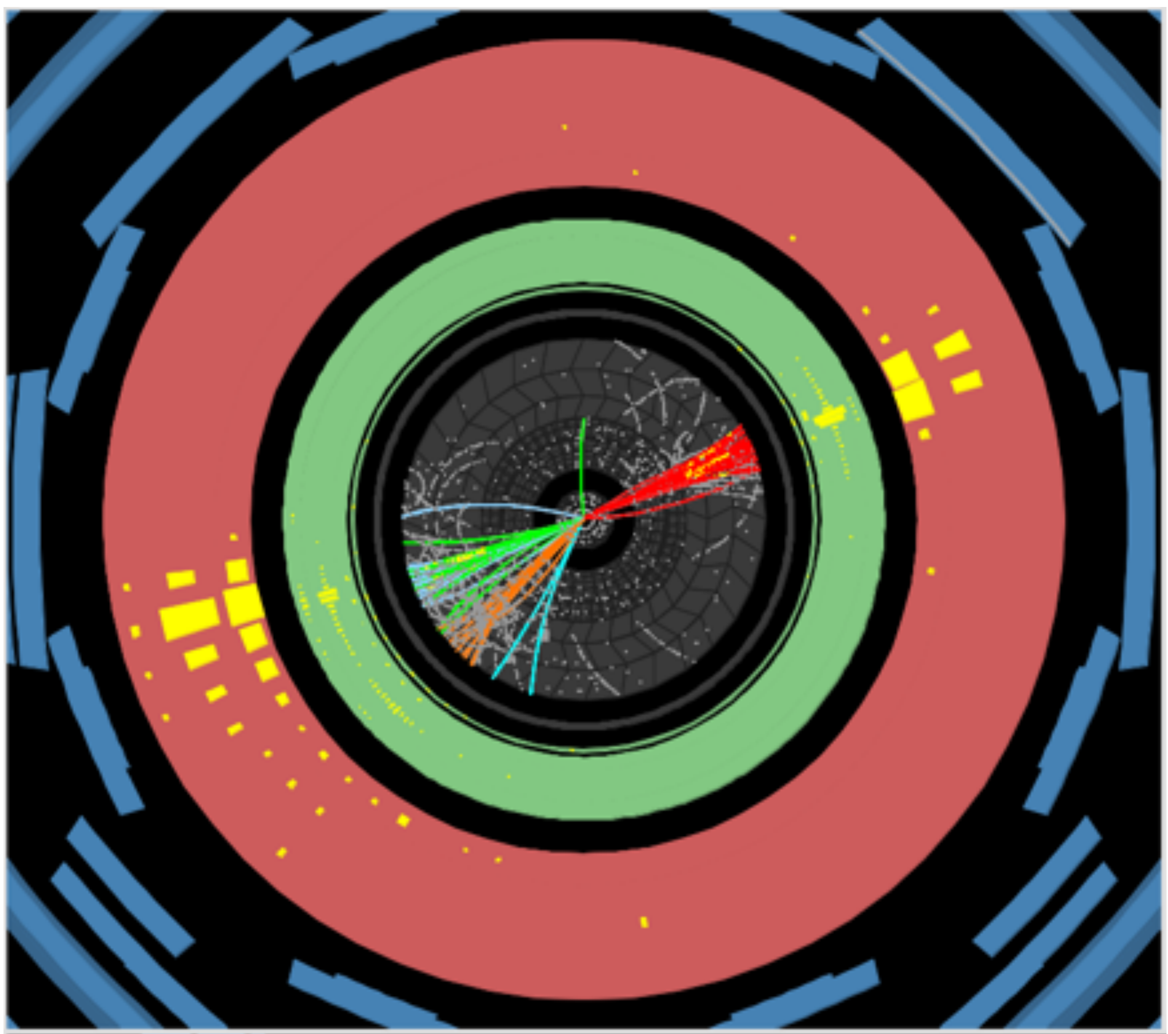
Particle Physics

Dr Victoria Martin, Spring Semester 2013

Lecture 9: Quantum Chromodynamics (QCD)



- ★Colour charge and symmetry
- ★Gluons
- ★QCD Feynman Rules
- ★$q\bar{q} \to q\bar{q}$ scattering
- ★QCD potential

Symmetries in Particle Physics

- The EM, Weak and Strong forces all display a property known as Gauge Symmetry.
- In QM, a symmetry is present if **physical observables** (e.g. cross section, decay widths) are invariant under the following change in the wavefunction:

$$\psi \rightarrow \psi' = \hat{U}\psi$$

- e.g. in electromagnetism, the physical observable fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are independent of the value of the EM potential, $\boldsymbol{A_\mu}$:

$$A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu \chi \qquad A_\mu = (V, \vec{A}) \text{ with } \vec{B} = \vec{\nabla} \times \vec{A}$$

- The conditions on $\boldsymbol{U}$ are that $\boldsymbol{U}$ is unitary, and commutes with the Hamiltonian:

$$\hat{U}^\dagger \hat{U} = \mathbf{1} \qquad [\hat{U}, \hat{H}] = 0$$

- e.g. for EM, $\hat{U} = e^{i\phi}$ where $\boldsymbol{\phi}$ is an arbitrary phase: $\psi \rightarrow \psi' = e^{i\phi}\psi$

Symmetries in QED

- Instead of a global phase transformation $e^{i\phi}$ imagine a local phase transformation, where the phase $\phi \sim q\,\chi$ is a function of x^μ: $\chi(x^\mu)$.
    - q is a constant (will be electric charge)

$$\psi \to \psi' = \hat{U}\psi = e^{iq\chi(x^\mu)}\psi$$

- Substitute into Dirac Equation $(i\gamma^\mu\partial_\mu - m)\psi = 0$

$$\begin{aligned}
(i\gamma^\mu\partial_\mu - m)\psi' &= 0 \\
(i\gamma^\mu\partial_\mu - m)e^{iq\chi(x)}\psi &= 0 \\
i\gamma^\mu(e^{iq\chi(x)}\partial_\mu\psi + iq\partial_\mu\chi\,\psi) - me^{iq\chi(x)}\psi &= 0 \\
(i\gamma^\mu\partial_\mu - m)e^{iq\chi(x)}\psi - q\gamma^\mu\partial_\mu\chi\,\psi &= 0
\end{aligned}$$

- An interaction term $-q\gamma^\mu\partial_\mu\chi\psi$ term appears in the Dirac Equation.
- To cancel this, modify the Dirac Equation for **interacting** fermions:

$$(i\gamma^\mu\partial_\mu + iqA_\mu - m)\psi = 0$$

- With A^μ transforming as:
- $A_\mu \to A'_\mu = A_\mu - \partial_\mu\chi$ to cancel interaction term

Gauge Symmetry in QED & QCD

- Demanding that QED is invariant by a local phase shift:

$$\psi \rightarrow \psi' = \hat{U}\psi = e^{iq\chi(x^\mu)}\psi$$

- Tells us that fermions interact with the photon field as:

$$q\gamma^\mu A_\mu \psi$$

- This invariance of QED under the local phase shift status is know as a **local U(1) gauge symmetry**.

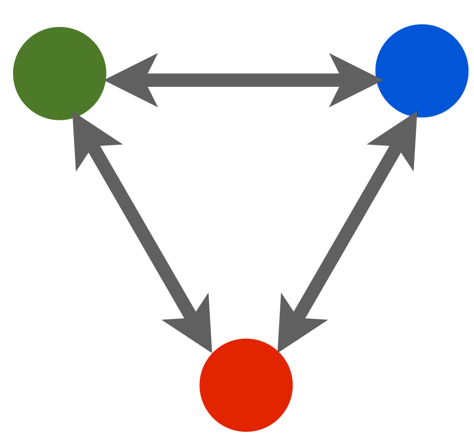
- Today we will see the consequences of a symmetry in QCD, but with a different symmetry, known as **SU(3).**
  - ➡ QCD exhibits a local SU(3) gauge symmetry.

Colour Charge

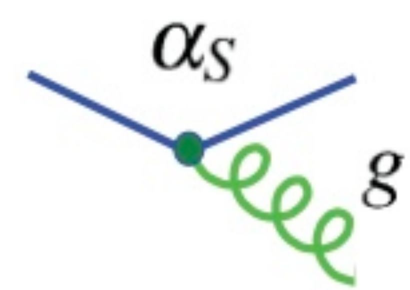
- Each quark carries a colour charge: red, blue or green.
- The coupling strength is the same for all three colours colours.
- To describe a quark, use a spinor **plus** a colour column vector:

$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad g = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- Mathematically, this is described by an SU(3) symmetry. QCD interactions are invariant under SU(3) rotations in colour space:



- Gluons responsible for exchanging momentum and colour between quarks.



- Each gluon contains colour and anti-colour.
- Naively expect nine gluons:
  $r\bar{r}$ $r\bar{b}$ $r\bar{g}$ $b\bar{r}$ $b\bar{b}$ $b\bar{g}$ $g\bar{r}$ $g\bar{b}$ $g\bar{g}$
- However gluons are described by the generators of the SU(3) group, giving eight linear colour–anti-colour combinations of these

Eight Gluons

- The Gell-Mann matrices describe the allowed colour configurations of gluons. (The Gell-Mann matrices are the *generators* of the SU(3) symmetry.)

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Each gluon is described by:

$$g^i = \begin{pmatrix} \mathrm{r} & \mathrm{b} & \mathrm{g} \end{pmatrix} \lambda^i \begin{pmatrix} \bar{\mathrm{r}} \\ \bar{\mathrm{b}} \\ \bar{\mathrm{g}} \end{pmatrix}$$

$$g^1 = \frac{1}{\sqrt{2}}(\mathrm{r\bar{b}} + \mathrm{b\bar{r}}) \quad g^2 = \frac{i}{\sqrt{2}}(\mathrm{r\bar{b}} - \mathrm{b\bar{r}}) \quad g^3 = \frac{1}{\sqrt{2}}(\mathrm{r\bar{r}} - \mathrm{b\bar{b}})$$

$$g^4 = \frac{1}{\sqrt{2}}(\mathrm{r\bar{g}} + \mathrm{g\bar{r}}) \quad g^5 = \frac{i}{\sqrt{2}}(\mathrm{r\bar{g}} - \mathrm{g\bar{r}}) \quad g^6 = \frac{1}{\sqrt{2}}(\mathrm{b\bar{g}} + \mathrm{g\bar{b}})$$

$$g^7 = \frac{i}{\sqrt{2}}(\mathrm{b\bar{g}} - \mathrm{g\bar{b}}) \quad g^8 = \frac{1}{\sqrt{6}}(\mathrm{r\bar{r}} + \mathrm{b\bar{b}} - 2\mathrm{g\bar{g}})$$

Feynman Rules for QCD

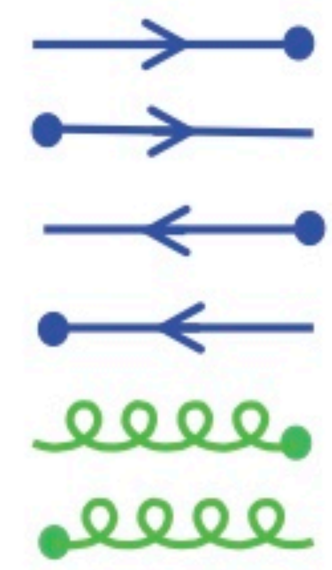
- **External Lines**

spin 1/2:

incoming quark	$u(p)$	
outgoing quark	$\overline{u}(p)$	
incoming anti-quark	$\overline{v}(p)$	
outgoing anti-quark	$v(p)$	

spin 1:

incoming gluon	$\varepsilon^{\mu}(p)$	
outgoing gluon	$\varepsilon^{\mu}(p)^{*}$	



- **Internal Lines (propagators)**

spin 1 gluon

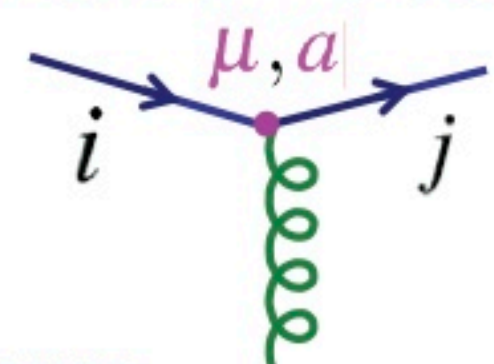
$$\frac{g_{\mu\nu}}{q^2}\delta^{ab}$$

a, b = 1,2,…,8 are gluon colour indices

- **Vertex Factors**

spin 1/2 quark

$$g_s \frac{1}{2}\lambda^{a}_{ji}\gamma^{\mu}$$



$$\alpha_S = \frac{g_S^2}{4\pi}$$

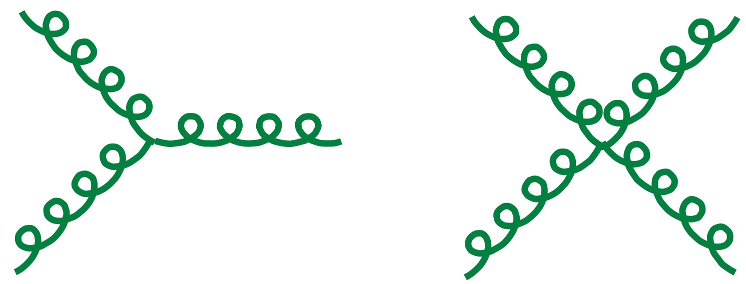
i, j = 1,2,3 are quark colours,

λ^a a = 1,2,..8 are the Gell-Mann SU(3) matrices

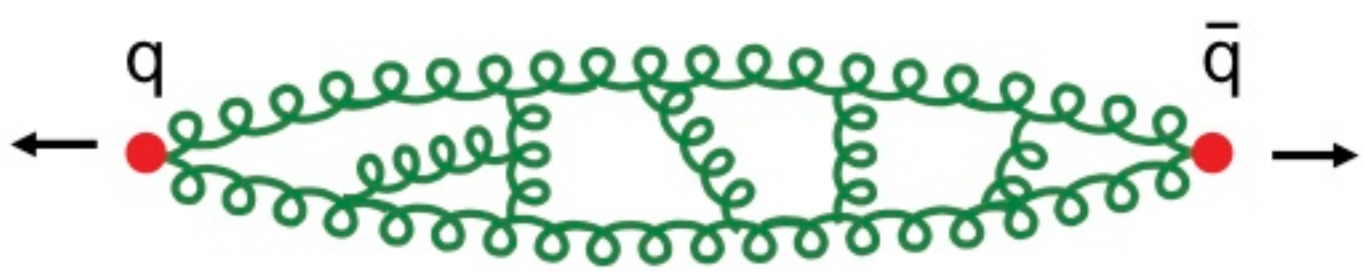
The λ^{a}_{ij} terms account for the quark colour

Gluon-Gluon Interactions

- Gluons also carry colour charge and can therefore self-interact.
- Two allowed possibilities:



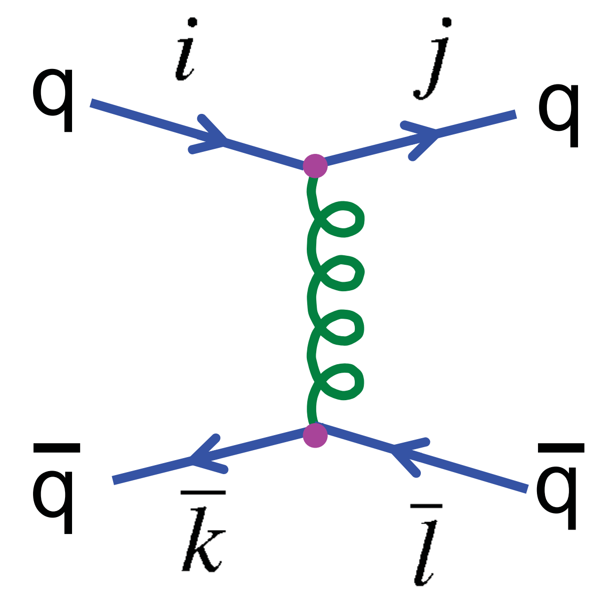
- Gluon interactions are believed to give rise to **colour confinement**
  - Try to separate an electron-positron pair $V_{\mathrm{QED}}(r) = -\frac{q_2\, q_1}{4\pi\epsilon_0 r} = -\frac{\alpha}{r}$
  - Try to separate an quark anti-quark pair $V_{\mathrm{QCD}}(r) \sim \lambda r$



- A gluon **flux tube** of interacting gluons is formed. Energy **~1 GeV/fm.**
- Gluon-gluon interactions are responsible for holding quarks in mesons and baryons.

$q\bar{q} \rightarrow q\bar{q}$ scattering

- To write down the matrix element, follow the fermion arrows backwards.
  - ➡ For the quark line $j \rightarrow i$: λ_{ji} term at vertex
  - ➡ For the antiquark line $k \rightarrow l$: λ_{kl} term at vertex



$$\mathcal{M} = \left[\bar{u}_j \frac{g_S}{2} \lambda^a_{ji} \gamma^\mu u_i\right] \frac{g^{\mu\nu}}{q^2} \delta^{ab} \left[\bar{v}_k \frac{g_S}{2} \lambda^b_{kl} \gamma^\nu v_l\right]$$

$$\mathcal{M} = \frac{g_S^2}{q^2} \frac{\lambda^a_{ji} \lambda^a_{kl}}{4} \left[\bar{u}_j \gamma^\mu u_i\right] \left[\bar{v}_k \gamma^\mu v_l\right]$$

- The matrix element looks very similar to electromagnetic scattering except $e \rightarrow g_S$, and the addition of the terms $\lambda^a_{ji} \lambda^a_{kl}/4$

- In the lowest order approximation, the dynamics of the $q\bar{q} \rightarrow q\bar{q}$ scattering is the same as electromagnetic $e^+e^- \rightarrow e^+e^-$ scattering.

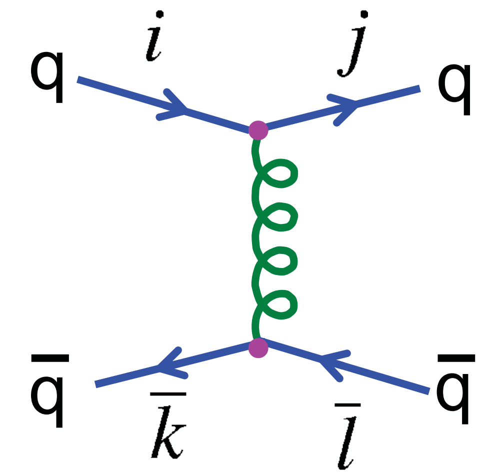
- Describe in terms of Coloumb-like potential $V_{q\bar{q}} = -\frac{f \alpha_S}{r}$

- The **colour factor** $f = \frac{1}{4}\lambda^a_{ji}\lambda^a_{kl} = \frac{1}{4}\sum_a \lambda^a_{ji}\lambda^a_{kl}$ is a sum over elements in the λ matrices.

Colour Factor for $q\bar{q}\rightarrow q\bar{q}$

- Need to calculate the **colour factor**

$$f = \frac{1}{4}\lambda^a_{ji}\lambda^a_{kl} = \frac{1}{4}\sum_a \lambda^a_{ji}\lambda^a_{kl}$$



- For the calculation we choose colours for q and $\bar{q}$. As the theory is invariant under rotations in colour space any choice of colours will give the same answer.

- Three colour options:
  1. i=1 k=$\bar{1}$ → j=1 l=$\bar{1}$ e.g. $r\bar{r}\rightarrow r\bar{r}$ $\quad f_1 = \frac{1}{4}\sum_a \lambda^a_{11}\lambda^a_{11}$
  2. i=1 k=$\bar{2}$ → j=1 l=$\bar{2}$ e.g. $r\bar{b}\rightarrow r\bar{b}$ $\quad f_2 = \frac{1}{4}\sum_a \lambda^a_{11}\lambda^a_{22}$
  3. i=1 k=$\bar{1}$ → j=2 l=$\bar{2}$ e.g. $r\bar{r}\rightarrow b\bar{b}$ $\quad f_3 = \frac{1}{4}\sum_a \lambda^a_{21}\lambda^a_{12}$

- Calculate option 2. The only matrices with non-zero element in the red-red (11) and blue-blue (22) elements are λ^3 and λ^8.

$$f_2 = \frac{1}{4}\sum_a \lambda^a_{11}\lambda^a_{22} = \frac{1}{4}[\lambda^3_{11}\lambda^3_{22} + \lambda^8_{11}\lambda^8_{22}] = \frac{1}{4}[(1)(-1) + (\frac{1}{\sqrt{3}})(\frac{1}{\sqrt{3}})] = -\frac{1}{6}$$

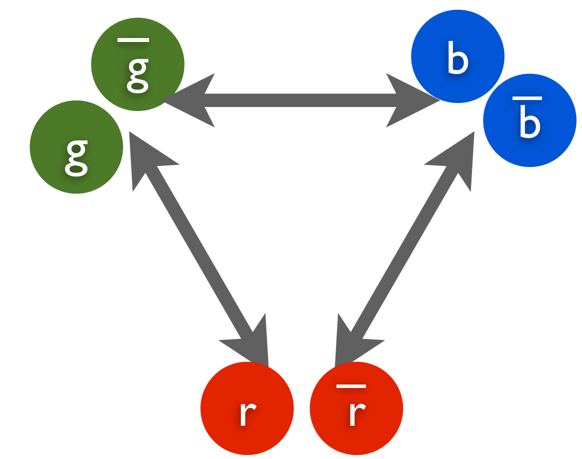
- Similarly,

$$f_1 = \frac{1}{4}\sum_a \lambda^a_{11}\lambda^a_{11} = \frac{1}{4}[\lambda^3_{11}\lambda^3_{11} + \lambda^8_{11}\lambda^8_{11}] = \frac{1}{4}[(1)(1) + (\frac{1}{\sqrt{3}})(\frac{1}{\sqrt{3}})] = \frac{1}{3}$$

$$f_3 = \frac{1}{4}\sum_a \lambda^a_{21}\lambda^a_{12} = \frac{1}{4}(\lambda^1_{21}\lambda^1_{12} + \lambda^2_{12}\lambda^2_{21}) = \frac{1}{4}[(-i)(i) + (1)(1)] = \frac{1}{2}$$

Colour Factor for Mesons

- Mesons are colourless $q\bar{q}$ states in a "colour singlet": $r\bar{r} + g\bar{g} + b\bar{b}$
- Calculate colour factor for $q\bar{q} \to q\bar{q}$ scattering in a meson.



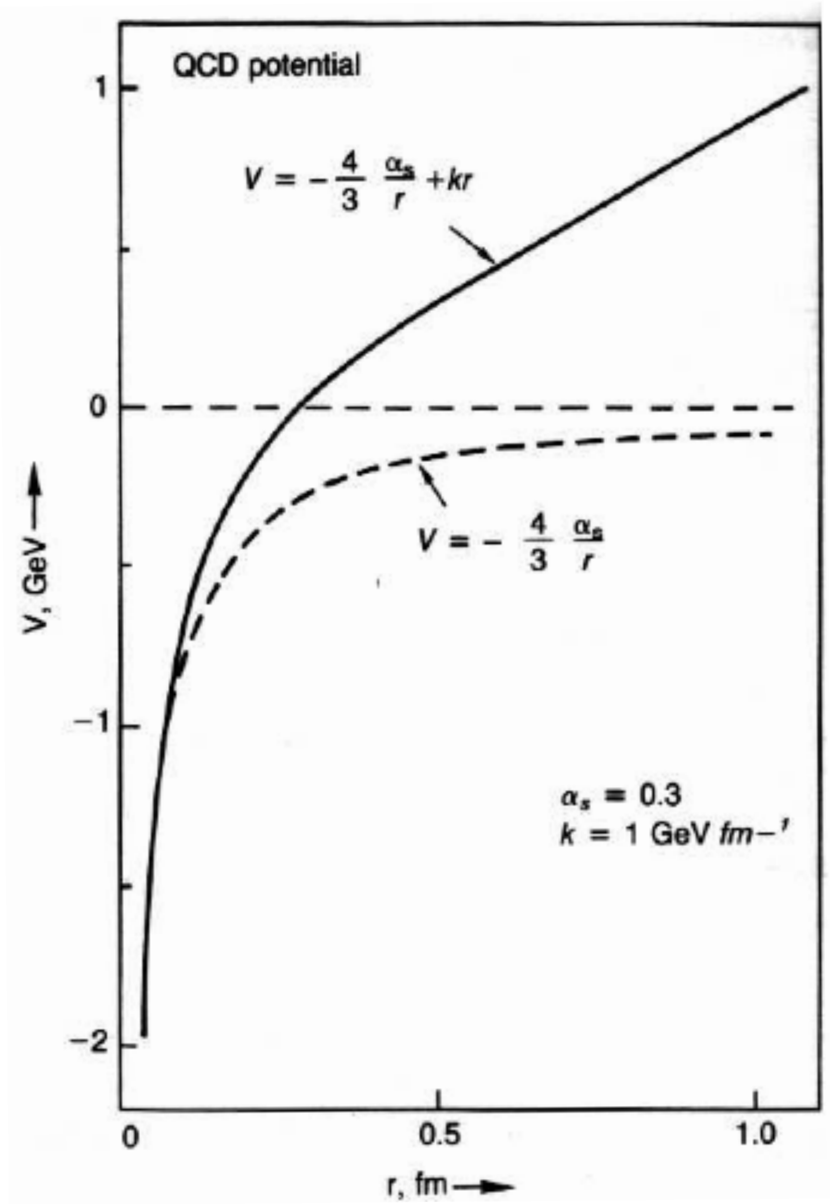
- Two possibilities for colour combinations:
  - ➡Quarks stay the same colour e.g. $r\bar{r} \to r\bar{r}$ $f_1 = \frac{1}{3}$
  - ➡Quarks change colour e.g. $r\bar{r} \to b\bar{b}$ and $r\bar{r} \to g\bar{g}$ each contributes $f_3 = \frac{1}{2}$

- Sum over all possible final states for $r\bar{r} \to q\bar{q}$ gives $f_r = \frac{1}{3} + \frac{1}{2} + \frac{1}{2} = 4/3$
- Average over all possible initial states, $r\bar{r}$, $g\bar{g}$, $b\bar{b}$:

$$f = \tfrac{1}{3}\,(r\bar{r} \to q\bar{q} + g\bar{g} \to q\bar{q} + b\bar{b} \to q\bar{q}) = \tfrac{1}{3}\,(4/3 + 4/3 + 4/3) = 4/3$$

- The colour factor for the $q\bar{q}$ interactions within a meson is **4/3**
  - The potential within a meson (to lowest order) is: $V_{q\bar{q}} = -\frac{4}{3}\frac{\alpha_S}{r}$

QCD Potential



- At large distances: gluon-gluon interactions

$$V_{\mathrm{QCD}}(r) \sim \lambda r$$

- At short distances: $\overline{q}q \rightarrow \overline{q}q$ scattering

$$V_{q\bar{q}} = -\frac{4}{3}\frac{\alpha_S}{r}$$

- Overall potential is:

$$V_{\mathrm{QCD}}(r) = -\frac{4}{3}\frac{\alpha_S}{r} + \lambda r$$

This model provides a good description of the bound states of heavy quarks:

- charmonium ($c\,\bar{c}$)
- bottomonium ($b\,\bar{b}$)