Workshop Questions

7 Questions

7.1 The sinc() function

State the expression for $\text{sinc}(x)$ in terms of $\sin(x)$, and prove that

$$\text{sinc}(0) = 1$$

Sketch the graph of

$$y = \text{sinc}(ax) \quad \text{and} \quad y = \text{sinc}^2(ax)$$

where a is a constant, and identify the locations of the zeros in each case.

7.2 Rectangular Aperture

Calculate the two dimensional Fourier transform of a rectangle of unit height and size a by b centered about the origin.

If $a = 5\,\text{mm}$ and $b = 1\,\text{mm}$ calculate the location of first zeros in the u and v direction. Sketch the real part of the Fourier transform. (`Maple` or `gnuplot` experts can make nice plots)

7.3 Gaussians

Calculate the Fourier Transform of a two-dimensional Gaussian given by,

$$f(x,y) = \exp\left(-\frac{r^2}{r_0^2}\right)$$

where $r^2 = x^2 + y^2$ and r_0 is the radius of the e^{-1} point.

You may use the standard mathematical identity that

$$\int_{-\infty}^{\infty} \exp(-bx^2)\; \exp(iax)\,\mathrm{d}x = \sqrt{\frac{\pi}{b}}\, \exp\left(-\frac{a^2}{4b}\right)$$

7.4 Differentials

Show, for a two dimensional function $f(x,y)$, that,

$$\mathcal{F}\left\{\frac{\partial f(x)}{\partial x}\right\} = \imath 2\pi u F(u)$$

and that

$$\mathcal{F}\left\{\nabla^2 f(x,y)\right\} = -(2\pi w)^2 F(u,v)$$

where $w^2 = u^2 + v^2$.

7.5 Delta Functions

Use one of the analytic definitions of the δ-function to show that

$$\mathcal{F}\{\delta(x)\} = 1$$

7.6 Sines and Cosines

Given the shifting property of the δ-function, begin:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,\mathrm{d}x = f(a)$$

then show that:

$$\mathcal{F}\{\delta(x-a)\} = \exp(\imath 2\pi a u)$$

Use this, or otherwise, to calculate

$$\mathcal{F}\{\cos(x)\} \quad \& \quad \mathcal{F}\{\sin(x)\}$$

7.7 Comb Function

Calculate the Fourier Transform of a one-dimensional infinte row of delta functions each separated a.

Consider the 3-dimensional case, and compare your result the reciprocal lattice of a simple cubic structure. (this example assumes that you are taking Solid State Physics).

7.8 Convolution Theorm

Prove the Convolution Theorm that if

$$g(x) = f(x) \odot h(x)$$

then we have that

$$G(u) = F(u)H(u)$$

where $F(u) = \mathcal{F}\{f(x)\}$ etc.

The Convolution is frequently described as *Fold-Shift-Multiply-Add*. Explain this be means of sketch diagrams in one-dimension.

7.9 Correlation Theorm

Prove the Correlation Theorm that if

$$c(x) = f(x) \otimes h(x)$$

then

$$C(u) = F(u)H^*(u)$$

and also that

$$h(x) \otimes f(x) = c^*(-x)$$

Show how the Correlation of two images is sometimes called "template-matching".

7.10 Auto-Correlation

Calculate the *Autocorrelation* of a two-dimensional square of side a centred on the origin. Use *Maple* of *gnuplot* to produce a three-dimensional plot of this function.

Hence calculate the two dimensional Fourier transform of the function

$$\begin{aligned} h(x,y) &= \left(1 - \frac{|x|}{a}\right)\left(1 - \frac{|y|}{b}\right) \quad |x| < a \text{ and } |y| < b \\ &= 0 \quad \text{else} \end{aligned}$$