Workshop Questions

7 Questions

7.1 The sinc() function

State the expression for $\mathrm{sinc}(x)$ in terms of $\sin(x)$, and prove that

$$\mathrm{sinc}(0) = 1$$

Sketch the graph of

$$y = \mathrm{sinc}(ax) \quad \text{and} \quad y = \mathrm{sinc}^2(ax)$$

where a is a constant, and identify the locations of the zeros in each case.

Solution

The definition of $\mathrm{sinc}(x)$ is

$$\mathrm{sinc}(x) = \frac{\sin(x)}{x}$$

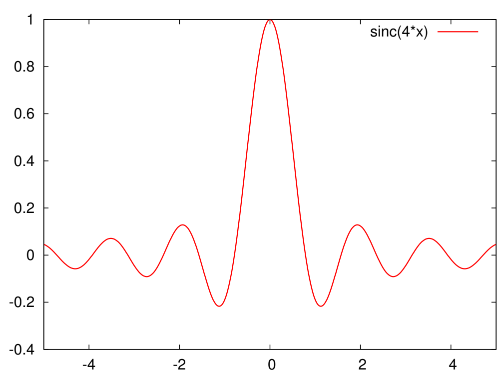
To find the value as $x \to 0$ take the Taylor expansion about $x = 0$ to get,

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

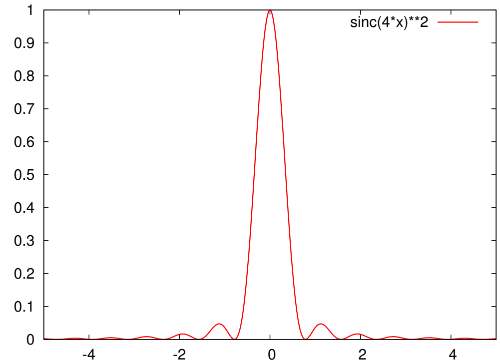
so we have that

$$\mathrm{sinc}(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots$$

so, when $x = 0$ then $\mathrm{sinc}(0) = 1$ as expected.
Sketch of $\mathrm{sinc}(ax)$ when $a = 4$



and sketch of $\mathrm{sinc}^2(ax)$ when $a = 4$



Both functions have zero in the same place, when $ax = \pm n\pi$, so at

$$x_n = \pm \frac{n\pi}{a} \quad n = 1, 2, \ldots$$

Note the **larger** a the closer the zero are together.

7.2 Rectangular Aperture

Calculate the two dimensional Fourier transform of a rectangle of unit height and size a by b centered about the origin.

If $a = 5$ mm and $b = 1$ mm calculate the location of first zeros in the u and v direction. Sketch the real part of the Fourier transform. (`Maple` or `gnuplot` experts can make nice plots)

Solution

We can express a rectangle of size $a \times b$ by:

$$\begin{aligned} f(x,y) &= 1 \quad |x| < a/2 \text{ and } |y| < b/2 \\ &= 0 \quad \text{else} \end{aligned}$$

the Fourier Transform is given by:

$$F(u,v) = \iint f(x,y) \, \exp(-\imath 2\pi(ux + vy)) \, \mathrm{d}x \, \mathrm{d}y$$

which can then be written as:

$$F(u,v) = \int_{-b/2}^{b/2} \left[\int_{-a/2}^{a/2} \exp(-\imath 2\pi(ux + vy)) \, \mathrm{d}x \right] \mathrm{d}y$$

Noting that the exp() term is separable, this can be written as

$$F(u,v) = \int_{-b/2}^{b/2} \exp(-\imath 2\pi vy) \, \mathrm{d}y \int_{-a/2}^{a/2} \exp(-\imath 2\pi ux) \, \mathrm{d}x$$

Look at one of the integrals, and we get,

$$\begin{aligned} \int_{-a/2}^{a/2} \exp(-\imath 2\pi ux) \, \mathrm{d}x &= \frac{1}{-\imath 2\pi u} [\exp(-\imath 2\pi ux]_{-a/2}^{a/2} \\ &= \frac{-\imath}{2\pi u} [\exp(\imath \pi au) - \exp(-\imath \pi ua)] \\ &= \frac{\sin(\pi au)}{\pi u} \\ &= a \operatorname{sinc}(\pi au) \end{aligned}$$

The other integral is of exactly the same form, so that the Fourier transform of the rectangle is:

$$F(u,v) = ab\,\mathrm{sinc}(\pi a u)\,\mathrm{sinc}(\pi b v)$$

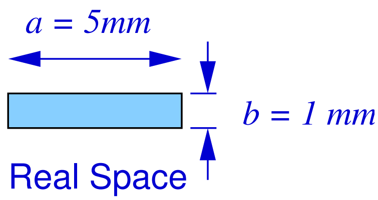
The zero of this function occur at:

$$\begin{aligned} u_n &= \pm\frac{n}{a} \quad \text{for } n = 1,2,3,\ldots \\ v_m &= \pm\frac{m}{b} \quad \text{for } m = 1,2,3,\ldots \end{aligned}$$

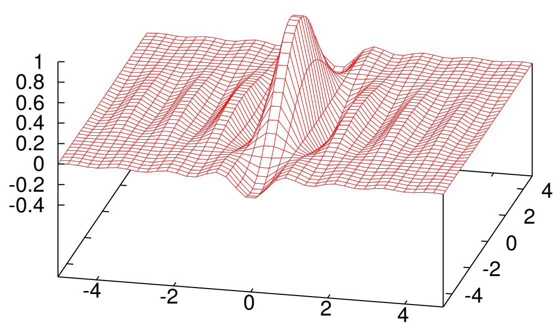
which if $a = 5\,\mathrm{mm}$ and $b = 1\,\mathrm{mm}$ then

$$\begin{aligned} u_n &= 0.2\,\mathrm{mm}^{-1}, 0.4\,\mathrm{mm}^{-1}, 0.6\,\mathrm{mm}^{-1}, \ldots \\ v_n &= 1\,\mathrm{mm}^{-1}, 2\,\mathrm{mm}^{-1}, 3\,\mathrm{mm}^{-1}, \ldots \end{aligned}$$

In diagrams we get,



so in Fourier space we get a three-Dimensional plot plot of



Note that the *long/thin* shape of the rectangle Fourier Transforms to *tall/thin* structures in the Fourier Transform.

7.3 Gaussians

Calculate the Fourier Transform of a two-dimensional Gaussian given by,

$$f(x,y) = \exp\left(-\frac{r^2}{r_0^2}\right)$$

where $r^2 = x^2 + y^2$ and r_0 is the radius of the e^{-1} point.
You may use the standard mathematical identity that

$$\int_{-\infty}^{\infty} \exp(-bx^2)\, \exp(iax)\, \mathrm{d}x = \sqrt{\frac{\pi}{b}}\, \exp\left(-\frac{a^2}{4b}\right)$$

Solution

The Fourier Transform is given by:

$$F(u,v) = \iint \exp\left(-\frac{(x^2+y^2)}{r_0^2}\right) \exp\left(-\imath 2\pi(ux+vy)\right) \mathrm{d}x\mathrm{d}y$$

Since the Gaussian *and* the Fourier kernel are separable, this can be written as

$$F(u,v) = \int \exp\left(-\frac{x^2}{r_0^2}\right) \exp(-\imath 2\pi ux)\mathrm{d}x \int \exp\left(-\frac{y^2}{r_0^2}\right) \exp(-\imath 2\pi vy)\mathrm{d}y$$

so we need only evaluate one integral.
Noting the result given that

$$\int_{-\infty}^{\infty} \exp(-bx^2)\exp(iax)\mathrm{d}x = \sqrt{\frac{\pi}{b}}\exp\left(-\frac{a^2}{4b}\right)$$

See "Mathematical Handbook", M.R. Spiegel, McGraw-Hill, Page 98, Definite Integral 15.73. The given identity is actually,

$$\int_{0}^{\infty} \exp(-bx^2)\cos(ax)\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{b}}\exp\left(-\frac{a^2}{4b}\right)$$

but this can be extended to the $\infty \to \infty$ exp() integral required by noting that the cos() is symmetric so $-\infty \to \infty$ integral is *double* the $0 \to \infty$ integral and that sin() is anti-symmetric so the imaginary part of the integral from $\infty \to \infty$ is zero.
Then if be let $b = 1/r_0^2$ and $a = 2\pi u$, then

$$\int \exp\left(-\frac{x^2}{r_0^2}\right) \exp(-\imath 2\pi ux)\mathrm{d}x = \frac{\sqrt{\pi}}{r_0}\exp\left(-\pi^2 r_0^2 u^2\right)$$

which is also a Gaussian.
Key Result: The Fourier Transform of a *Gaussian* is a *Gaussian*. It is the *only function* that is its own Fourier Transform.
Exactly the same expression for the y integral, so we get that

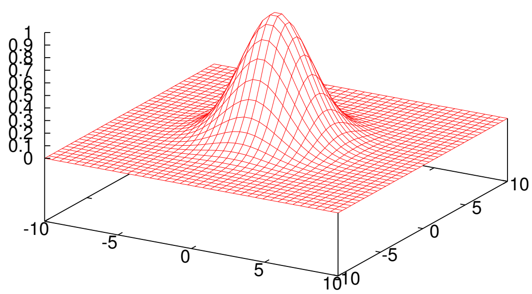
$$F(u,v) = \frac{\pi}{r_0^2}\exp\left(-\pi^2 r_0^2 (u^2+v^2)\right)$$

which is more conveniently written as:

$$F(u,v) = \frac{\pi}{r_0^2}\exp\left(-\frac{w^2}{w_0^2}\right)$$

where $w^2 = u^2 + v^2$ and $w_0 = 1/\pi r_0$, which is a circular Gaussian radius with e^{-1} point at w_0. So the Fourier Transform of a Gaussian is a Gaussian of reciprocal width. Or more simply, as a *wide Gaussian* Fourier Transform for give a *narrow Gaussian* and vice versa.

General shape of two dimensional Gaussian with $r_0 = 3$ is given by



7.4 Differentials

Show, for a two dimensional function $f(x,y)$, that,

$$\mathcal{F}\left\{\frac{\partial f(x)}{\partial x}\right\} = \imath 2\pi u F(u)$$

and that

$$\mathcal{F}\left\{\nabla^2 f(x,y)\right\} = -(2\pi w)^2 F(u,v)$$

where $w^2 = u^2 + v^2$.

Solution

If $F(u)$ is the Fourier Transform of $f(x)$ then

$$f(x) = \mathcal{F}^{-1}\{F(u)\}$$

which we can write out as:

$$f(x) = \int F(u)\exp(\imath 2\pi u x)\,\mathrm{d}u$$

take differential of both sides,

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \int \imath 2\pi u F(u)\exp(\imath 2\pi u x)\,\mathrm{d}u$$

showing that the left side is

$$\mathcal{F}^{-1}\{\imath 2\pi u F(u)\}$$

take the forward Fourier transform of each side to give:

$$\mathcal{F}\left\{\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right\} = \imath 2\pi u\, F(u)$$

as required.

In two dimensions we have the a similar result that:

$$\mathcal{F}\left\{\frac{\partial f(x,y)}{\partial x}\right\} = \imath 2\pi u F(u,v)$$

and that:

$$\mathcal{F}\left\{\frac{\partial f(x,y)}{\partial y}\right\} = \imath 2\pi v F(u,v)$$

the second order differentials are thus:

$$\mathcal{F}\left\{\frac{\partial^2 f(x,y)}{\partial x^2}\right\} = -(2\pi u)^2 F(u,v)$$

and that:

$$\mathcal{F}\left\{\frac{\partial^2 f(x,y)}{\partial y^2}\right\} = -(2\pi v)^2 F(u,v)$$

The Laplacian,

$$\nabla^2 f(x,y) = \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}$$

so noting that the Fourier Transform is a linear relation, we get that

$$\mathcal{F}\left\{\nabla^2 f(x,y)\right\} = -(2\pi)^2(u^2+v^2)F(u,v) = -(2\pi w)^2 F(u,v)$$

as required.

The result that taking the Laplacian in real space is equivalent to multiplying by a parabolic term in Fourier space is used in image processing to detect edges.

7.5 Delta Functions

Use one of the analytic definitions of the δ-function to show that

$$\mathcal{F}\left\{\delta(x)\right\} = 1$$

Solution

Take the Top-Hat definition of the δ-function, with

$$\Delta_\varepsilon(x) = \frac{1}{\varepsilon}\Pi\left(\frac{x}{\varepsilon}\right)$$

The Gaussian definition is similar, but the sinc() *definition is a bit more difficult since it Fourier Transform to give a* $\Pi()$ *which is not actually analytic.*

The Fourier Transform is given by:

$$\mathcal{F}\left\{\Delta_\varepsilon(x)\right\} = \frac{1}{\varepsilon}\int_{-\varepsilon/2}^{\varepsilon/2} \exp(-\imath 2\pi u x)\,\mathrm{d}x$$

which we can integrate to give

$$\frac{1}{\varepsilon}\frac{\imath}{2\pi u}\left[\exp(-\imath 2\pi u x\right]_{-\varepsilon/2}^{\varepsilon/2}$$

which gives

$$\frac{1}{\varepsilon}\frac{\imath}{2\pi u}\left[\exp(-\imath\pi\varepsilon u) - \exp(\imath\pi\varepsilon u)\right]$$

which we can then write as:

$$\frac{1}{\varepsilon}\frac{\imath}{2\pi u} - 2\imath\sin(\pi\varepsilon u)$$

which is then just

$$\frac{\sin(\pi\varepsilon u)}{\pi\varepsilon u} = \mathrm{sinc}(\pi\varepsilon u)$$

now we have from question 1, we that $\mathrm{sinc}(0) = 1$, so we have that

$$\lim_{\varepsilon\to 0}\mathcal{F}\left\{\Delta_\varepsilon(x)\right\} = 1$$

as expected.

7.6 Sines and Cosines

Given the shifting property of the δ-function, begin:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,\mathrm{d}x = f(a)$$

then show that:

$$\mathcal{F}\left\{\delta(x-a)\right\} = \exp(\imath 2\pi a u)$$

Use this, or otherwise, to calculate

$$\mathcal{F}\left\{\cos(x)\right\} \quad \& \quad \mathcal{F}\left\{\sin(x)\right\}$$

Solution

We can write,

$$\mathcal{F}\left\{\delta(x-a)\right\} = \int \delta(x-a)\exp(-\imath 2\pi u x)\,\mathrm{d}x$$

then the shifting property gives that this is just the value of exp() as $x = a$, so that

$$\mathcal{F}\left\{\delta(x-a)\right\} = \exp(-\imath 2\pi a u)$$

take the inverse Fourier Transform of both sides gives,

$$\mathcal{F}^{-1}\left\{\exp(-\imath 2\pi a u)\right\} = \delta(x-a)$$

noting that the difference between a *forward* and *inverse* Fourier Transform is just a $-$ sign, then,

$$\mathcal{F}\left\{\exp(\imath 2\pi a u)\right\} = \delta(x-a)$$

let $a = 1/2\pi$ and interchange x and u to give,

$$\mathcal{F}\left\{\exp(\imath x)\right\} = \delta\left(u - \frac{1}{2\pi}\right)$$

Now, noting that the Fourier Transform is linear, then:

$$\mathcal{F}\left\{\cos(x)\right\} = \frac{1}{2}\left[\mathcal{F}\left\{\exp(\imath x)\right\} + \mathcal{F}\left\{\exp(-\imath x)\right\}\right] = \frac{1}{2}\left[\delta\left(u - \frac{1}{2\pi}\right) + \delta\left(u + \frac{1}{2\pi}\right)\right]$$

and similarly,

$$\mathcal{F}\left\{\sin(x)\right\} = \frac{1}{2\imath}\left[\mathcal{F}\left\{\exp(\imath x)\right\} - \mathcal{F}\left\{\exp(-\imath x)\right\}\right] = \frac{1}{2\imath}\left[\delta\left(u - \frac{1}{2\pi}\right) - \delta\left(u + \frac{1}{2\pi}\right)\right]$$

so cos() and sin() Fourier transform to give a single frequency, as expected.

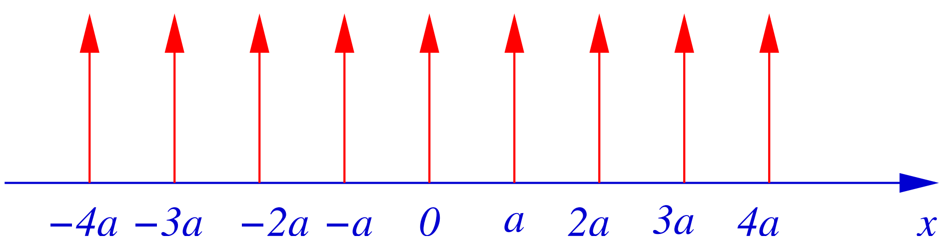
7.7 Comb Function

Calculate the Fourier Transform of a one-dimensional infinte row of delta functions each separated a.

Consider the 3-dimensional case, and compare your result the reciprocal lattice of a simple cubic structure. (this example assumes that you are taking Solid State Physics).

Solution

An infinite row of δ-functions separated by a,



This is known as a δ-Comb, which can be written as

$$\text{Comb}(x) = \sum_{j=-\infty}^{\infty} \delta(x - ja)$$

Note that the Fourier Transform of one δ-function is

$$\mathcal{F}\left\{\delta(x-a)\right\} = \exp(-\imath 2\pi a u)$$

so noting that the Fourier Transform is linear, then the FT of the Comb function is

$$F(u) = \sum_{j=-\infty}^{\infty} \exp(-\imath 2\pi j a u)$$

Now we have that,

$$\exp(-\imath 2\pi j a u) = 1 \quad \text{if } 2\pi a u = 2n\pi$$

so that then

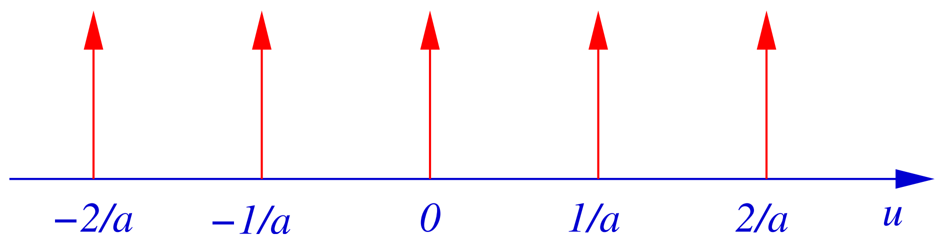
$$u = \frac{n}{a} \quad \Rightarrow \quad \exp(-\imath 2\pi jau) = 1 \quad \forall j$$

to the Fourier Transform

$$\begin{aligned} F(u) &\rightarrow \infty \quad \text{when } u = n/a \text{ (In Phase)} \\ &\rightarrow 0 \quad \text{when } u \neq n/a \text{ (Out of Phase)} \end{aligned}$$

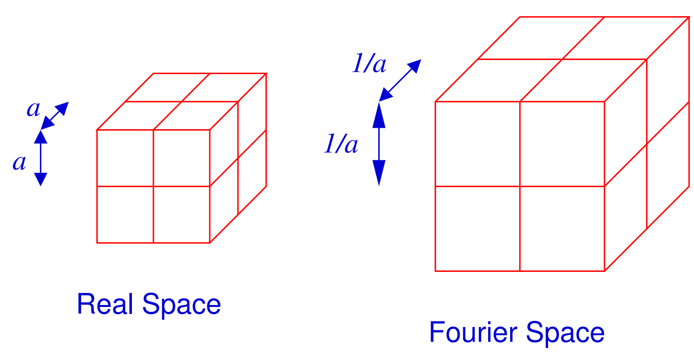
so $F(u)$ is also a δ-Comb with spacing $1/a$, which can be written as

$$F(u) = \sum_{j=-\infty}^{\infty} \delta\left(u - \frac{j}{a}\right)$$



Note the *reciprocal* relation between real and Fourier Space.

In the case of a three-dimensional lattice the atomic locations are given by three-dimensional δ-function (points in three-dimensional space). So a simple cubic lattice is just a three-dimensional δ-Comb. The Fourier transform is separable so we can take the Fourier Transform in each dimension separately. Each transform takes a δ-Comb in real space to a reciprocally spaced δ-Comb is Fourier space, so that the Fourier Transform of a simple cubic lattice is a simple cubic structure in Fourier space.



In solid state physics, the *Reciprocal Lattice* is a Three-Dimensional Fourier Transform of the real space lattice.

All the other lattice structures can be Fourier transformed by considering breaking the structure down into δ-Combs, for example the Fourier transforms of **fcc** is **bcc** etc.

7.8 Convolution Theorm

Prove the Convolution Theorm that if

$$g(x) = f(x) \odot h(x)$$

then we have that

$$G(u) = F(u)\,H(u)$$

where $F(u) = \mathcal{F}\{f(x)\}$ etc.
The Convolution is frequently described as *Fold-Shift-Multiply-Add*. Explain this be means of sketch diagrams in one-dimension.

Solution

Convolution is defined as

$$g(x) = f(x) \odot h(x) = \int_{-\infty}^{\infty} f(s)\,h(x-s)\mathrm{d}s$$

Now take the Fourier Transform of both sides, to get

$$\int g(x)\exp(-\imath 2\pi ux)\mathrm{d}x = \int \left[\int_{-\infty}^{\infty} f(s)\,h(x-s)\mathrm{d}s\right]\exp(-\imath 2\pi ux)\mathrm{d}x$$

The Fourier Transform is linear, so the order of integration does not matter, so we get

$$G(u) = \iint f(s)\,h(x-s)\exp(-\imath 2\pi ux)\mathrm{d}s\mathrm{d}x$$

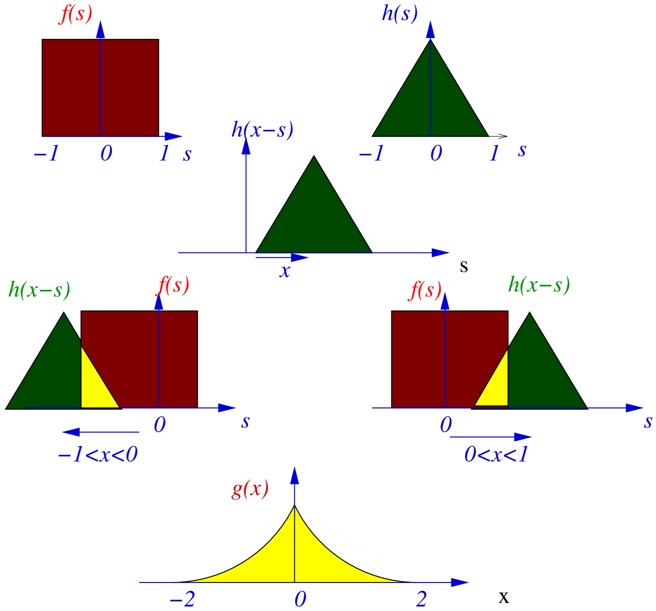
Now let $t = x - s$ so we get

$$\begin{aligned} G(u) &= \iint f(s)\,h(t)\exp\left(-\imath 2\pi u(s+t)\right)\mathrm{d}s\mathrm{d}t \\ &= \int f(s)\exp(-\imath 2\pi us)\mathrm{d}s \int h(t)\exp(-\imath 2\pi ut)\mathrm{d}t \\ &= F(u)\,H(u) \end{aligned}$$

where $F(u) = \mathcal{F}\{f(x)\}$, $H(u) = \mathcal{F}\{h(x)\}$ and $G(u) = \mathcal{F}\{g(x)\}$.
Convolution can be described as:

1. *Fold and Shift* $h(x-s)$ can be interperated at taking function $h(s)$, "Folding" to get $h(-s)$ and "Shifting" by distance x to get $h(x-s)$.
2. *Multiply* the folded and shifted function $h(x-s)$ is $\times$ by $f(s)$
3. *Add* up the area of overlap, (or more formally integrate).

This can be consider diagrammatically as:



7.9 Correlation Theorm

Prove the Correlation Theorm that if

$$c(x) = f(x) \otimes h(x)$$

then

$$C(u) = F(u)H^*(u)$$

and also that

$$h(x) \otimes f(x) = c^*(-x)$$

Show how the Correlation of two images is sometimes called "template-matching".

Solution

Correlation is defined as

$$c(x) = f(x) \otimes h(x) = \int_{-\infty}^{\infty} f(s)\, h^*(s-x)\mathrm{d}s$$

Now take the Fourier Transform of both sides, to get

$$\int c(x)\exp(-\imath 2\pi u x)\mathrm{d}x = \int \left[\int_{-\infty}^{\infty} f(s)\, h^*(s-x)\mathrm{d}s\right]\exp(-\imath 2\pi u x)\mathrm{d}x$$

The Fourier Transform is linear, so the order of integration does not matter, so we get

$$C(u) = \iint f(s)\, h^*(s-x)\exp(-\imath 2\pi u x)\mathrm{d}s\mathrm{d}x$$

Now let $t = s - x$ so we get

$$C(u) \quad = \quad \iint f(s)\, h^*(t)\exp(-\imath 2\pi u(s-t))\,\mathrm{d}s\mathrm{d}t$$

$$= \int f(s)\exp(-\imath 2\pi us)\mathrm{d}s \int h(t)^{*}\exp(\imath 2\pi ut)\mathrm{d}t$$
$$= \int f(s)\exp(-\imath 2\pi us)\mathrm{d}s \left[\int h(t)\exp(-\imath 2\pi ut)\mathrm{d}t\right]^{*}$$
$$= F(u)H^{*}(u)$$

where $F(u) = \mathcal{F}\{f(x)\}$, $H(u) = \mathcal{F}\{h(x)\}$ and $C(u) = \mathcal{F}\{c(x)\}$.

Correlation is very similar to convolution *except* the second function is not *folded* and the direction of the shift is reversed. We have that,

$$c(x) = \int_{-\infty}^{\infty} f(s)h^{*}(s-x)\mathrm{d}s$$

so taking the complex conjugate, we have that

$$c^{*}(x) = \int_{-\infty}^{\infty} f^{*}(s)h(s-x)\mathrm{d}s$$

Now let $t = s - x$, so we

$$c^{*}(x) = \int_{-\infty}^{\infty} f^{*}(t+x)h(t)\mathrm{d}t$$

then letting $y = -x$, we get

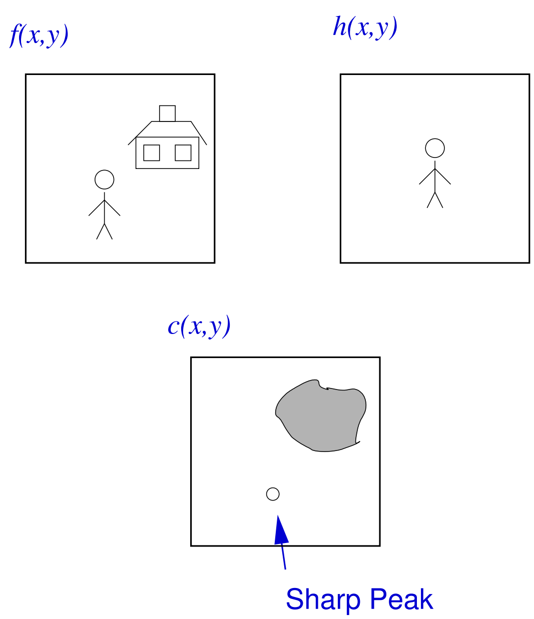
$$c^{*}(-y) = \int_{-\infty}^{\infty} h(t)f^{*}(t-y)\mathrm{d}t$$

so that we have that

$$h(x) \otimes f(x) = c^{*}(-x)$$

In most cases the functions $f(x)$ and $h(x)$ will be real, so the complex conjugation does not matter, but the reversal does.

Take a input scene $f(x,y)$ and a target function $h(x,y)$, then the correlation is formed by taking a shifted version of $h(x,y)$, the target, and placing over the input scene $f(x,y)$.



When *man* is located over *house* there is a poor match, so that multiplying and summing the overlap will give a an indistinct *blob*.

When *man* is located over *man* there is a good match so that multiplying and summing will give a sharp *peak*. This height of the *peak* will give the degree at match between the *target* and the *input scene* and the location of the peak will give the location of the *target*. The correlation is thus the degree of match between the target and the input scene, and hence *template-matching*.

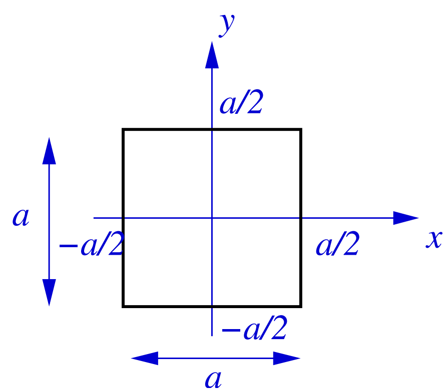
7.10 Auto-Correlation

Calculate the *Autocorrelation* of a two-dimensional square of side a centred on the origin. Use *Maple* of *gnuplot* to produce a three-dimensional plot of this function.
Hence calculate the two dimensional Fourier transform of the function

$$\begin{aligned} h(x,y) &= \left(1-\frac{|x|}{a}\right)\left(1-\frac{|y|}{b}\right) \quad |x|<a \text{ and } |y|<b \\ &= 0 \quad \text{else} \end{aligned}$$

Solution

Take a square of size $a \times a$ centred about the origin we have



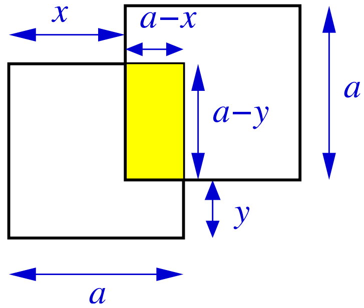
The autocorrelation is given, mathematically, by

$$a(x,y) = \iint f(s,t)\, f^*(s-x,t-y)\,\mathrm{d}s\mathrm{d}t$$

in this case $f(x,y)$ is real.
Physically this means:

1. Shift $f(s,t)$ by amount (x,y).
2. Multiply with the unshifted version.
3. Integrate over the area of overlap.

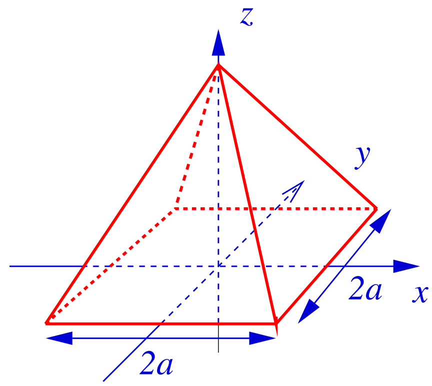
So if we shift by (x,y) we get



so the *Area of Overlap* is

$$a(x,y) = (a-|x|)\,(a-|y|) = a^2\left(1-\frac{|x|}{a}\right)\left(1-\frac{|y|}{a}\right)$$

This is a square pyramid with base $2a \times 2a$



Note that the autocorrelation is **twice** the size of of the original square.
The function $h(x,y)$ is the Normalised Autocorrelation, so that

$$h(x,y) = \frac{a(x,y)}{a^2}$$

The Fourier Transform of $h(x,y)$ is

$$H(u,v) = \mathcal{F}\left\{h(x,y)\right\} = \frac{1}{a^2}\mathcal{F}\left\{a(x,y)\right\} = \frac{1}{a^2}A(u,v)$$

The autocorrelation theorem gives at

$$\begin{aligned} a(x,y) &= f(x,y) \otimes f(x,y) \\ A(u,v) &= |F(u,v)|^2 \end{aligned}$$

Now $f(x,y)$ is a square of size $a \times a$, so from Question 2 we have that.

$$F(u,v) = a^2 \mathrm{sinc}(\pi a u)\mathrm{sinc}(\pi a v)$$

So we have that

$$A(u,v) = a^4 \mathrm{sinc}^2(\pi a u)\mathrm{sinc}^2(\pi a v)$$

and so the required Fourier Transform

$$H(u,v) = a^2 \mathrm{sinc}^2(\pi a u)\mathrm{sinc}^2(\pi a v)$$

This is much easier than trying to form the direct Fourier Transform.