4 Symmetry Conditions

When we take the the Fourier Transform of a *real* function, for example a one-dimensional sound signal or a two-dimensional image we obtain a *complex* Fourier Transform. This Fourier Transform has special symmetry properties that are essential when calculating and/or manipulating Fourier Transforms. This section it of the booklet is mainly aimed at the DIGITAL IMAGE ANALYSIS and THEORY OF IMAGE PROCESSING courses that make extensive use of these symmetry conditions.

4.1 One-Dimensional Symmetry

Firstly consider the case of a one dimensional real function $f(x)$, with a Fourier transform of $F(u)$. Since $f(x)$ is real then from previous we can write

$$F(u) = F_r(u) + \imath F_\imath(u)$$

where the real and imaginary parts are given by the cosine and sine transforms to be

$$\begin{aligned} F_r(u) &= \int f(x)\cos(2\pi ux)\,\mathrm{d}x \\ F_\imath(u) &= -\int f(x)\sin(2\pi ux)\,\mathrm{d}x \end{aligned} \tag{1}$$

now $\cos()$ is a symmetric function and $\sin()$ is an anti-symmetric function, as shown in figure 1, so that:

$$\begin{array}{ll} F_r(u) & \text{is Symmetric} \\ F_\imath(u) & \text{is Anti-symmetric} \end{array}$$

which can be written out explicitly as,

$$\begin{aligned} F_r(u) &= F_r(-u) \\ F_\imath(u) &= -F_\imath(-u) \end{aligned} \tag{2}$$

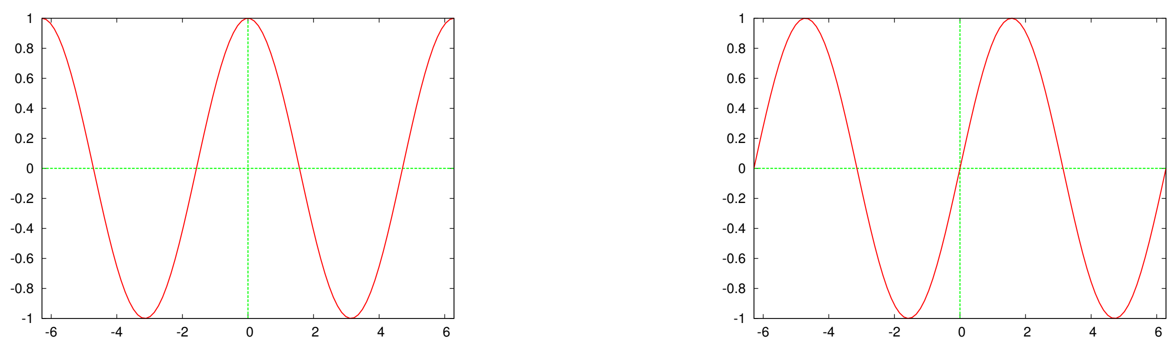

Figure 1: Symmetry properties of cos() and sin() functions

The *power spectrum* is given by

$$|F(u)|^2 = F_r(u)^2 + F_\imath(u)^2$$

so that if the real and imaginary parts obey the symmetry property given in equation (38), then clearly the *power spectrum* is also symmetric with

$$|F(u)|^2 = |F(-u)|^2 \tag{3}$$

so when the power spectrum of a signal is calculated it is normal to display the signal from $0 \to u_{\max}$ and ignore the negative components.

4.2 Two-Dimensional Symmetry

In two dimensional we have a real image $f(x,y)$, and then as above the Fourier transform of this image can be written as,

$$F(u,v) = F_r(u,v) + \imath F_\imath(u,v) \tag{4}$$

where after expansion of the exp() functions into cos() and sin() functions we get that

$$F_r(u,v) = \iint f(x,y)\,[\cos(2\pi ux)\,\cos(2\pi vy) - \sin(2\pi ux)\,\sin(2\pi vy)]\;\mathrm{d}x\,\mathrm{d}y$$

and that;

$$F_\imath(u,v) = \iint f(x,y)\,[\cos(2\pi ux)\,\sin(2\pi vy) + \sin(2\pi ux)\,\cos(2\pi vy)]\;\mathrm{d}x\,\mathrm{d}y$$

In this case the symmetry properties are more complicated, however we say that the real part is symmetric and the imaginary part is anti-symmetric, where in two dimensions the symmetry conditions are given by,

$$\begin{aligned} F_r(u,v) &= F_r(-u,-v) \\ F_r(-u,v) &= F_r(u,-v) \end{aligned} \tag{5}$$

for the real part of the Fourier transform, and

$$\begin{aligned} F_\imath(u,v) &= -F_\imath(-u,-v) \\ F_\imath(-u,v) &= -F_\imath(u,-v) \end{aligned} \tag{6}$$

for the imaginary part. Similarly the two dimensional *power spectrum* is also symmetric, with

$$\begin{aligned} |F(u,v)|^2 &= |F(-u,-v)|^2 \\ |F(-u,v)|^2 &= |F(u,-v)|^2 \end{aligned} \tag{7}$$

This symmetry condition is shown schematically in figure 2, which shows a series of symmetric points.

These symmetry properties has a major significance in the digital calculation of Fourier transforms and the design of digital filters, which is discussed in greater detail in the relevant courses.

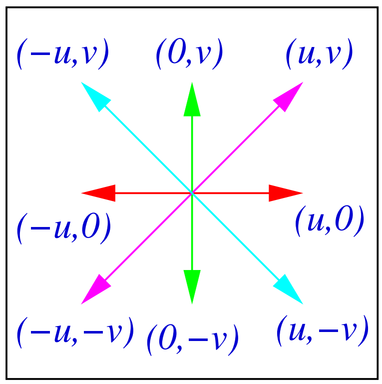

Figure 2: Symmetry in two dimensions