Fourier Theory

Aim: The lecture covers the Fourier Theory as detailed in FOURIER TRANSFORM, (WHAT YOU NEED TO KNOW).

Contents:

1. Introduction and Notation
2. The Fourier Transform and its Properties
3. The Dirac Delta Function
4. Symmetry Conditions of Fourier Transforms
5. Convolution and Correlation
6. Summary

Notation

The notation maintained throughout will be:

$$
\begin{array}{lcl}
x, y & \longrightarrow & \text{Real Space co-ordinates} \\
u, v & \longrightarrow & \text{Frequency Space co-ordinates}
\end{array}
$$

and lower case functions $f(x)$, being a real space function and upper case functions (*eg* $F(u)$), being the corresponding Fourier transform, thus:

$$
\begin{aligned}
F(u) &= \mathcal{F}\{f(x)\} \\
f(x) &= \mathcal{F}^{-1}\{F(u)\}
\end{aligned}
$$

where $\mathcal{F}\{\}$ is the Fourier Transform operator.

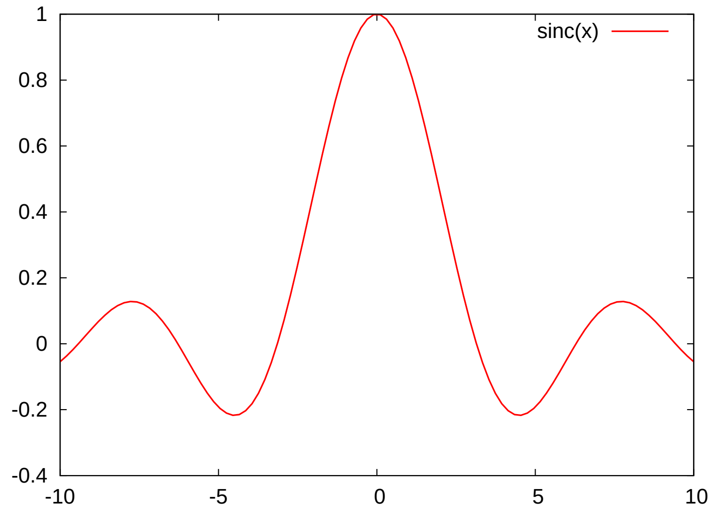
The character $\imath$ will be used to denote $\sqrt{-1}$.

Special Functions

Two special functions,

The $\mathrm{sinc}()$ Function:

$$\mathrm{sinc}(x) = \frac{\sin(x)}{x}$$

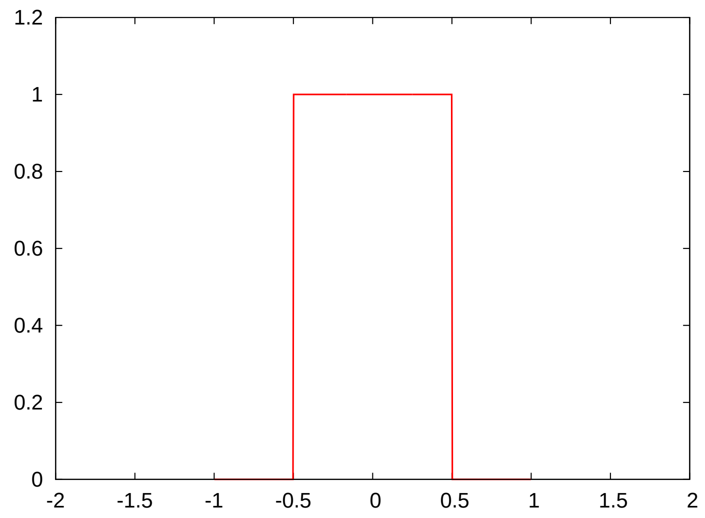


Special Functions

The Top-Hat Function:

$$\begin{aligned} \Pi(x) &= 1 \quad \text{for } |x| \leq 1/2 \\ &= 0 \quad \text{else} \end{aligned}$$

begin of unit height and width centered about $x = 0$,



The Fourier Transform

For dimensional continuous function, $f(x)$

$$F(u) = \int_{-\infty}^{\infty} f(x) \exp(-\imath 2\pi u x) \, \mathrm{d}x$$

with inverse Fourier transform by;

$$f(x) = \int_{-\infty}^{\infty} F(u) \exp(\imath 2\pi u x) \, \mathrm{d}u$$

If $f(x)$ is a *real* signal

$$F(u) = F_r(u) + \imath F_\imath(u)$$

where we have,

$$\begin{aligned} F_r(u) &= \int_{-\infty}^{\infty} f(x) \cos(2\pi u x) \, \mathrm{d}x \\ F_\imath(u) &= -\int_{-\infty}^{\infty} f(x) \sin(2\pi u x) \, \mathrm{d}x \end{aligned}$$

Desomposition of $f(x)$ into $\cos()$ and $\sin()$ terms.

The u variable is interpreted as a frequency, so. $f(x)$ is a sound signal x in seconds.

$F(u)$ is its frequency spectrum with u measured in Hertz (s^{-1}).

Properties of the Fourier Transform

The Fourier transform has a range of useful properties, some of which are listed below.

Linearity: The Fourier transform is a linear operation, so.

$$\mathcal{F}\left\{af(x)+bg(x)\right\} = aF(u)+bG(u)$$

Central when describing *linear* systems.

Complex Conjugate: The Fourier transform *Complex Conjugate* of a function is given by

$$\mathcal{F}\left\{f^*(x)\right\} = F^*(-u)$$

where $F(u)$ is the Fourier transform of $f(x)$.

Forward and Inverse: We have that

$$\mathcal{F}\left\{F(u)\right\} = f(-x)$$

apply Fourier transform twice, get a spatial reversal.

Similarly with inverse Fourier transform

$$\mathcal{F}^{-1}\left\{f(x)\right\} = F(-u)$$

so that the Fourier and inverse Fourier transforms differ only by a sign.

Properties of the Fourier Transform I

Differentials: The Fourier transform of the derivative is

$$\mathcal{F}\left\{\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right\} = \imath 2\pi u F(u)$$

and the second derivative is given by

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -(2\pi u)^2 F(u)$$

Used frequently in signal and image processing.

Power Spectrum: The *Power Spectrum* is modulus square of the Fourier transform

$$P(u) = |F(u)|^2$$

. This can be interpreted as the *power* of the frequency components.

Any function and its Fourier transform obey the condition that

$$\int_{-\infty}^{\infty} |f(x)|^2 \,\mathrm{d}x = \int_{-\infty}^{\infty} |F(u)|^2 \,\mathrm{d}u$$

which is frequently known as *Parseval's Theorem*.

Power in real and Fourier space in the same.

Two Dimensional Fourier Transform

Two dimensional Fourier transform of a function $f(x,y)$ by,

$$F(u,v) = \iint f(x,y)\exp(-\imath 2\pi(ux+vy))\,\mathrm{d}x\,\mathrm{d}y$$

with the inverse Fourier transform defined by;

$$f(x,y) = \iint F(u,v)\exp(\imath 2\pi(ux+vy))\,\mathrm{d}u\,\mathrm{d}v$$

Real function $f(x,y)$, the Fourier transform can be considered as the decomposition of a function into its sinusoidal components.

Note: x,y usually have dimensions of *length*.

Fourier space variables u,v dimensions of *inverse* length, called *Spatial Frequency*.

Clearly the derivatives then become

$$\mathcal{F}\left\{\frac{\partial f(x,y)}{\partial x}\right\} = \imath 2\pi u F(u,v) \quad \text{and} \quad \mathcal{F}\left\{\frac{\partial f(x,y)}{\partial y}\right\} = \imath 2\pi v F(u,v)$$

yielding the important result that,

$$\mathcal{F}\left\{\nabla^2 f(x,y)\right\} = -(2\pi w)^2 F(u,v) \quad \text{where } w^2 = u^2 + v^2$$

Two Dimensional Fourier transform II

Two dimensional Fourier Transform of a function is a separable operation.

$$F(u,v) = \int P(u,y) \exp(-\imath 2\pi v y) \, \mathrm{d}y$$

where

$$P(u,y) = \int f(x,y) \exp(-\imath 2\pi u x) \, \mathrm{d}x$$

where $P(u,y)$ is the Fourier Transform of $f(x,y)$ with respect to x only.

Special case when $f(x,y)$ also seperable, so that

$$f(x,y) = f_a(x) \, f_b(y)$$

then we have that

$$F(u,v) = F_a(u) \, F_b(v)$$

where

$$F_a(u) = \mathcal{F}\{f_a(x)\} \quad \text{and} \quad F_b(v) = \mathcal{F}\{f_b(y)\}$$

vastly simplifying the calculation.

The Three-Dimensional Fourier Transform

Three dimensional case we have a function $f(\vec{r})$ where $\vec{r} = (x, y, z)$, then the three-dimensional Fourier Transform

$$F(\vec{s}) = \iiint f(\vec{r}) \exp(-\imath 2\pi \vec{r}.\vec{s})\, \mathrm{d}\vec{r}$$

where $\vec{s} = (u, v, w)$ being the three reciprocal variables each with units length^{-1}.

Similarly the inverse Fourier Transform is given by

$$f(\vec{r}) = \iiint F(\vec{s}) \exp(\imath 2\pi \vec{r}.\vec{s})\, \mathrm{d}\vec{s}$$

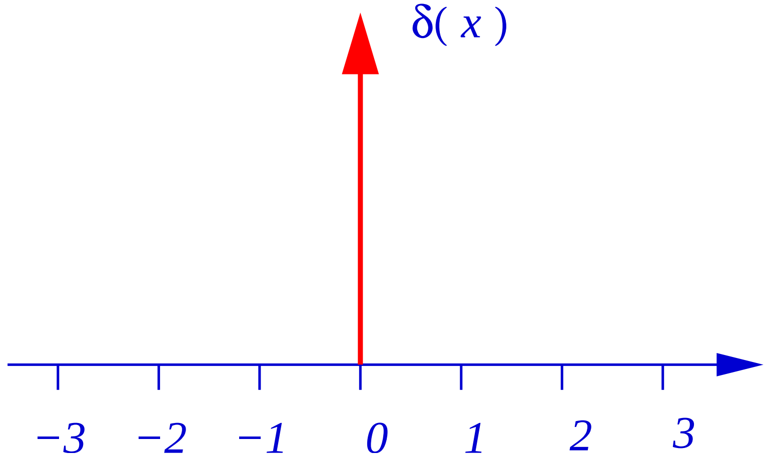
Used extensively in solid state physics where the three-dimensional Fourier Transform of a crystal structures is usually called *Reciprocal Space*.

Dirac Delta Function

Dirac Delta Function, which is somewhat abstractly defined as:

$$\begin{aligned} \delta(x) &= 0 \qquad \text{for } x \neq 0 \\ \int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x &= 1 \end{aligned}$$

"*tall*-and-*thin*" spike with unit area located at the origin,



not an "infinitely high" since it scales,

$$\int_{-\infty}^{\infty} a\,\delta(x)\,\mathrm{d}x = a$$

where a is a constant.

Dirac Delta Function II

There are a range of definitions in terms of *proper function*, are:

$$\Delta_\varepsilon(x) = \frac{1}{\varepsilon\sqrt{\pi}}\exp(\frac{-x^2}{\varepsilon^2})$$

$$\Delta_\varepsilon(x) = \frac{1}{\varepsilon}\Pi\left(\frac{x-\frac{1}{2}\varepsilon}{\varepsilon}\right)$$

$$\Delta_\varepsilon(x) = \frac{1}{\varepsilon}\mathrm{sinc}\left(\frac{x}{\varepsilon}\right)$$

all have the property that,

$$\int_{-\infty}^{\infty}\Delta_\varepsilon(x)\,\mathrm{d}x = 1 \quad \forall\varepsilon$$

and we may form the approximation that,

$$\delta(x) = \lim_{\varepsilon\to 0}\Delta_\varepsilon(x)$$

which can be interpreted as making any of the above approximations $\Delta_\varepsilon(x)$ a very "*tall*-and-*thin*" spike with unit area.

Dirac Delta Function III

In the field of optics and imaging useful to define *Two Dimensional Dirac Delta Function*

$$\begin{aligned} \delta(x,y) &= 0 \qquad \text{for } x \neq 0 \;\&\; y \neq 0 \\ \iint \delta(x,y)\,\mathrm{d}x\,\mathrm{d}y &= 1 \end{aligned}$$

which is the two dimensional version of the $\delta(x)$ function defined above, and in particular:

$$\delta(x,y) = \delta(x)\,\delta(y).$$

Can be considered as a single bright spot in the centre of the field of view, for example a single bright star viewed by a telescope.

Properties of the Dirac Delta Function

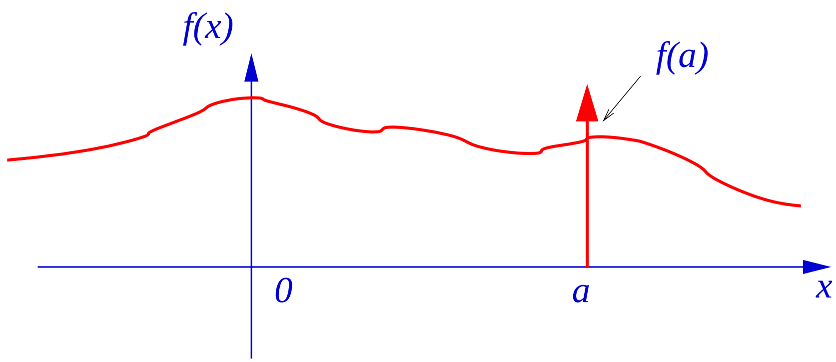
For a function $f(x)$ we have that

$$\int_{-\infty}^{\infty} \delta(x)\, f(x)\, \mathrm{d}x = f(0)$$

which is often taken as an alternative definition of the Delta function. Extended to the *Shifting Property* of

$$\int_{-\infty}^{\infty} \delta(x-a)\, f(x)\, \mathrm{d}x = f(a)$$

where $\delta(x-a)$ is just a δ-function located at $x = a$



In two dimensions, for a function $f(x,y)$, we have that,

$$\iint \delta(x-a, y-b)\, f(x,y)\, \mathrm{d}x\, \mathrm{d}y = f(a,b)$$

where $\delta(x-a, y-b)$ is a δ-function located at position a, b.

Properties of the Delta Function I

The Fourier transform by integration of the definition,

$$\mathcal{F}\{\delta(x)\} = \int_{-\infty}^{\infty} \delta(x) \exp(-\imath 2\pi u x)\, \mathrm{d}x = \exp(0) = 1$$

and then by the *Shifting Theorem* we get that,

$$\mathcal{F}\{\delta(x-a)\} = \exp(\imath 2\pi a u)$$

typically called a *phase ramp*.

Noted that the modulus squared is

$$|\mathcal{F}\{\delta(x-a)\}|^2 = |\exp(-\imath 2\pi a u)|^2 = 1$$

the power spectrum a *Delta Function* is a constant independent of its location.

Properties of the Delta Function II

Two *Delta Function* located at $\pm a$, then

$$\mathcal{F}\{\delta(x-a)+\delta(x+a)\} = \exp(\imath 2\pi a u) + \exp(-\imath 2\pi a u) = 2\cos(2\pi a u)$$

while if we have the *Delta Function* at $x = -a$ as negative,

$$\mathcal{F}\{\delta(x-a)-\delta(x+a)\} = \exp(\imath 2\pi a u) - \exp(-\imath 2\pi a u) = 2\imath\sin(2\pi a u)$$

So we get the two useful results that

$$\mathcal{F}\{\cos(2\pi a x)\} = \frac{1}{2}[\delta(u+a)+\delta(u-a)]$$

and that

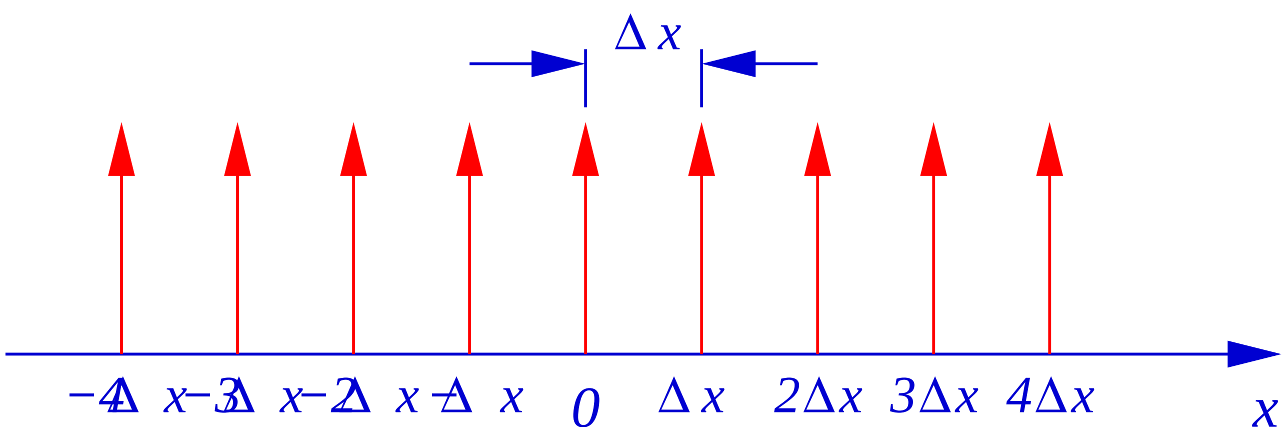
$$\mathcal{F}\{\sin(2\pi a x)\} = \frac{1}{2\imath}[\delta(u+a)-\delta(u-a)]$$

So that the Fourier transform of a cosine or sine function consists of a single frequency.

The Infinite Comb

Series of Delta functions at a regular spacing of Δx,, giving

$$\mathrm{Comb}_{\Delta x}(x) = \sum_{i=-\infty}^{\infty} \delta(x - i\Delta x).$$



Fourier transform is sum of the Fourier transforms of shifted Delta functions,

$$\mathcal{F}\left\{\mathrm{Comb}_{\Delta x}(x)\right\} = \sum_{i=-\infty}^{\infty} \exp(-\imath 2\pi i \Delta x u)$$

The Infinite Comb I

Now the exponential term,

$$\exp(-\imath 2\pi i \Delta x u) = 1 \quad \text{when } 2\pi\Delta x u = 2\pi n$$

so that:

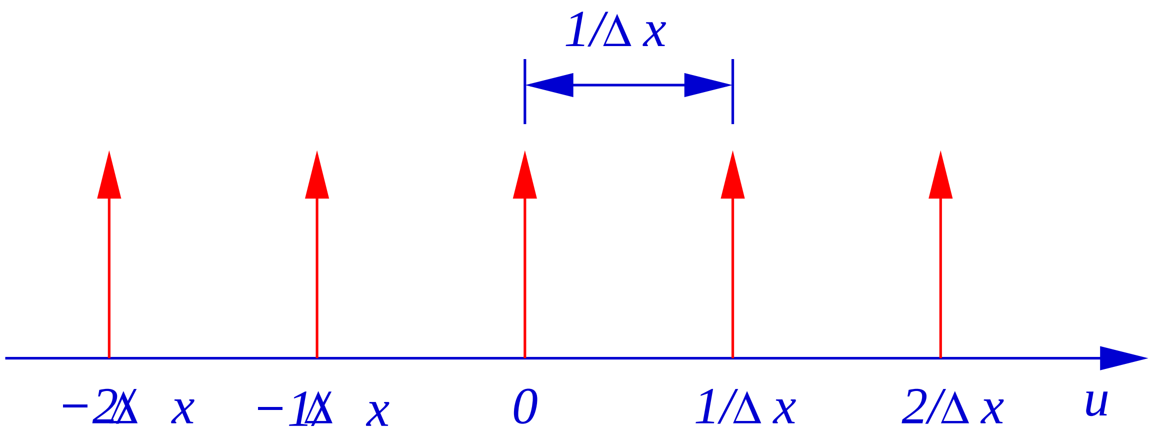
$$\begin{aligned}\sum_{i=-\infty}^{\infty} \exp(-\imath 2\pi i \Delta x u) &\to \infty \quad \text{when } u = \frac{n}{\Delta x} \\ &= 0 \quad \text{else}\end{aligned}$$

which is an infinite series of δ-function at a separation of $\Delta u = \frac{1}{\Delta x}$.

The Infinite Comb II

So that an *Infinite Comb* Fourier transforms to another *Infinite Comb*

$$\mathcal{F}\left\{\mathrm{Comb}_{\Delta x}(x)\right\} = \mathrm{Comb}_{\Delta u}(u) \quad \text{with } \Delta u = \frac{1}{\Delta x}$$



Symmetry Conditions

For a *real* function has a *complex* Fourier Transform.

This Fourier Transform has special symmetry properties that are essential when calculating and/or manipulating Fourier Transforms.

One-Dimensional Symmetry: Since $f(x)$ is real then,

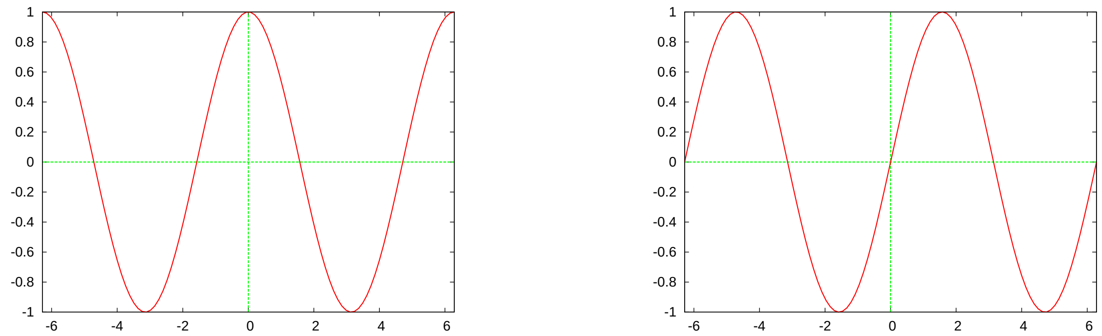
$$F(u) = F_r(u) + \imath F_\imath(u)$$

where we have

$$\begin{aligned} F_r(u) &= \int f(x)\cos(2\pi u x)\,\mathrm{d}x \\ F_\imath(u) &= -\int f(x)\sin(2\pi u x)\,\mathrm{d}x \end{aligned}$$

Symmetry Conditions I

now $\cos()$ is a symmetric function and $\sin()$ is an anti-symmetric



$F_r(u)$ is Symmetric and $F_\imath(u)$ is Anti-symmetric

which can be written out explicitly as,

$$F_r(u) = F_r(-u) \quad \text{and} \quad F_\imath(u) = -F_\imath(-u)$$

The *power spectrum* is given by

$$|F(u)|^2 = F_r(u)^2 + F_\imath(u)^2$$

then clearly the *power spectrum* is also symmetric with

$$|F(u)|^2 = |F(-u)|^2$$

so when the power spectrum of a signal is calculated it is normal to display the signal from $0 \rightarrow u_{\max}$ and ignore the negative components.

Symmetry Conditions III

Two-Dimensional Symmetry real fucntion $f(x,y)$, then

$$F(u,v) = F_r(u,v) + \imath F_\imath(u,v)$$

expand $\exp()$ functions into $\cos()$ and $\sin()$ we get that

$$F_r(u,v) = \iint f(x,y)\,[\cos(2\pi ux)\cos(2\pi vy) - \sin(2\pi ux)\sin(2\pi vy)]\,\mathrm{d}x\,\mathrm{d}y$$

$$F_\imath(u,v) = \iint f(x,y)\,[\cos(2\pi ux)\sin(2\pi vy) + \sin(2\pi ux)\cos(2\pi vy)]\,\mathrm{d}x\,\mathrm{d}y$$

real part is symmetric and the imaginary part is anti-symmetric,

$$\begin{aligned} F_r(u,v) &= F_r(-u,-v) \\ F_r(-u,v) &= F_r(u,-v) \end{aligned}$$

for the real part of the Fourier transform, and

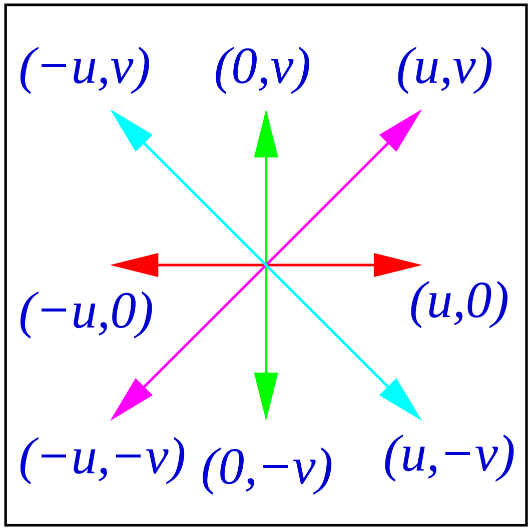
$$\begin{aligned} F_\imath(u,v) &= -F_\imath(-u,-v) \\ F_\imath(-u,v) &= -F_\imath(u,-v) \end{aligned}$$

for the imaginary part.

Symmetry Conditions IV

The *power spectrum* is also symmetric, with

$$
\begin{aligned}
|F(u,v)|^2 &= |F(-u,-v)|^2 \\
|F(-u,v)|^2 &= |F(u,-v)|^2
\end{aligned}
$$



Convolution of Two Functions

Convolution is central to Fourier theory.

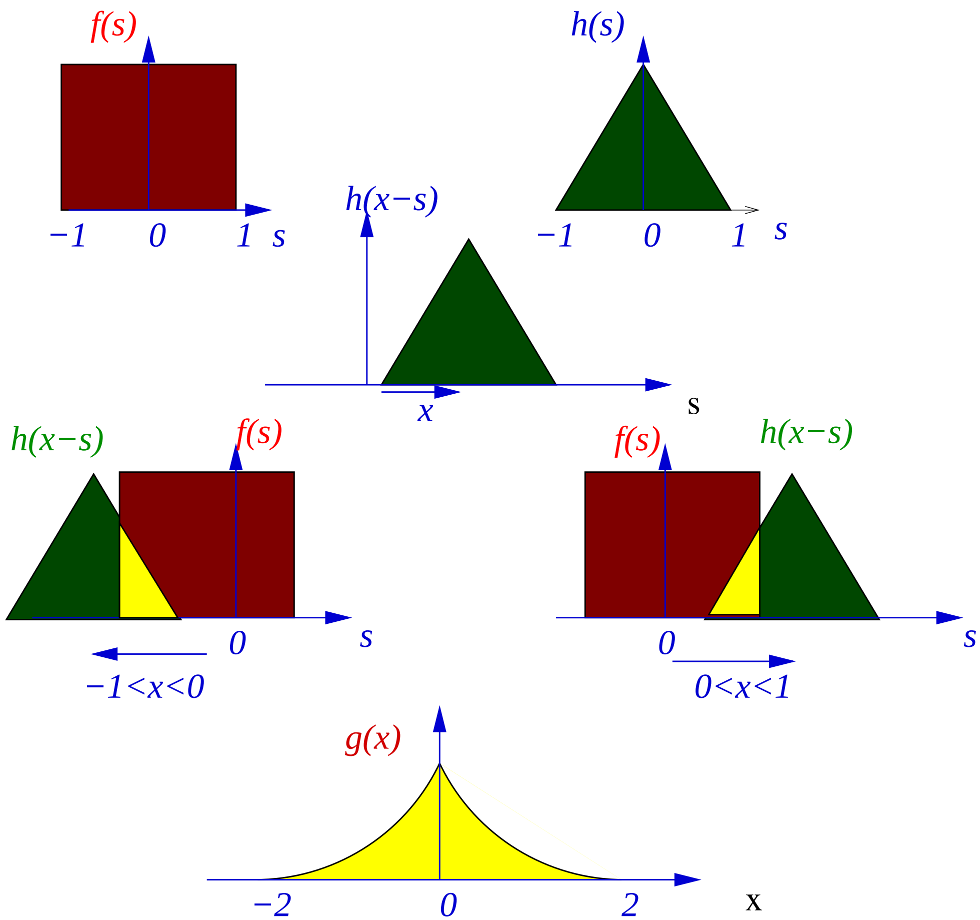
Convolution between two functions, $f(x)$ and $h(x)$ is defined as:

$$g(x) = f(x) \odot h(x) = \int_{-\infty}^{\infty} f(s)\, h(x-s)\, \mathrm{d}s$$

where s is a dummy variable of integration.

Convolution of Two Functions I

Area of overlap between the function $f(x)$ and the *spatially reversed version* of the function $h(x)$.



Convolution of Two Functions II

The *Convolution Theorem* is

$$G(u) = F(u)H(u)$$

where

$$\begin{aligned} G(u) &= \mathcal{F}\{g(x)\} \\ F(u) &= \mathcal{F}\{f(x)\} \\ H(u) &= \mathcal{F}\{h(x)\} \end{aligned}$$

This is the most important result here!

Simple Properties on Convolution:

Linear operation which is distributative, so that for three functions $f(x)$, $g(x)$ and $h(x)$

$$f(x) \odot (g(x) \odot h(x)) = (f(x) \odot g(x)) \odot h(x)$$

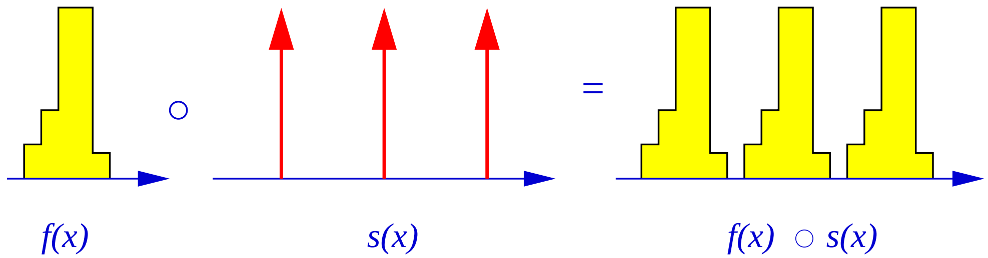
and commutative, so that

$$f(x) \odot h(x) = h(x) \odot f(x)$$

If $f(x)$ and $h(x)$ are of finite width, then extent (or “width”) of $g(x)$ is given by the sum of the widths the two functions.

Convolution with Comb

Convolution of a function $f(x)$ with a $\mathrm{Comb}(x)$ function results in replication of the function at the comb spacing.



(obvious from the *shift theorm*).

Which then Fourier Transforms to give

$$\mathcal{F}\{f(x)\odot s(x)\} = F(u)\,S(u)$$

where $S(u)$ is also on Comb or recriprocal spacing. This is fundamental to Sampling Theory.

Two Dimensional Convolution

Extension to two-dimensions is simple with,

$$g(x,y) = f(x,y) \odot h(x,y) = \int\!\!\int f(s,t)h(x-s,y-t)\,\mathrm{d}s\,\mathrm{d}t$$

which in the Fourier domain gives the important result that,

$$G(u,v) = F(u,v)\,H(u,v)$$

The most important implication of the *Convolution Theorem* is that,

Multiplication in Real Space	$\Longleftrightarrow$	Convolution in Fourier Space
Convolution in Real Space	$\Longleftrightarrow$	Multiplication in Fourier Space

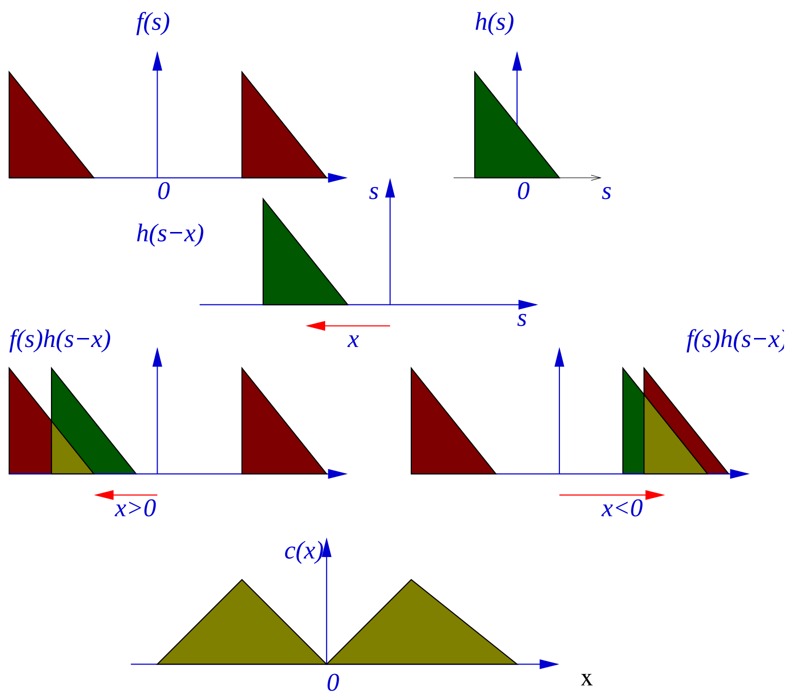
which is a **Key Result**, especially in optics.

Correlation of Two Functions

A closely related operation is *Correlation*. The *Correlation* between two function $f(x)$ and $h(x)$ is

$$c(x) = f(x) \otimes h(x) = \int_{-\infty}^{\infty} f(s)\, h^{*}(s-x)\, \mathrm{d}s$$

Note for real $h(x)$, different only by a $-$sign.



so second function is *not* reversed.

Correlation between Two Functions I

In the Fourier Domain the *Correlation Theorem* becomes

$$C(u) = F(u)\, H^*(u)$$

where

$$\begin{aligned} C(u) &= \mathcal{F}\{c(x)\} \\ F(u) &= \mathcal{F}\{f(x)\} \\ H(u) &= \mathcal{F}\{h(x)\} \end{aligned}$$

This which is distributative, but however is **not** commutative, since if

$$c(x) = f(x) \otimes h(x)$$

then we can show that

$$h(x) \otimes f(x) = c^*(-x)$$

In two dimensions we have the correlation between two functions given by

$$c(x,y) = f(x,y) \otimes h(x,y) = \iint f(s,t)\, h^*(s-x, t-y)\, \mathrm{d}s\, \mathrm{d}t$$

which in Fourier space gives,

$$C(u,v) = F(u,v)\, H^*(u,v)$$

Correlation is used in optics to to characterise the incoherent optical properties of a system and in digital imaging as a measure of the “similarity” between two images.

Autocorrelation

Special case of *correlation* of a function with is self is *Autocorrelation* being,

$$a(x,y) = f(x,y) \otimes f(x,y)$$

so that in Fourier space we have,

$$A(u,v) = F(u,v)\,F^*(u,v) = |F(u,v)|^2$$

which is the *Power Spectrum* of the function $f(x,y)$.

Autocorrelation of a function is given by the *Inverse Fourier Transform* of the *Power Spectrum*

$$a(x,y) = \mathcal{F}^{-1}\left\{|F(u,v)|^2\right\}$$

In this case the *correlation* must be commutative, so we have that

$$a^*(-x,-y) = a(x,y)$$

If $f(x,y)$ is real, then $a(x,y)$ is real, so is symmetric.

If we detect the *Power Spectrum* of object, we cannot reform the object, only is *autocorrelation*.

Summary

This lecture covers the practical aspects of the Fourier Transform at its applications mainly in two-dimensional systems.

1. Introduction and Notation
2. The Fourier Transform and its Properties
3. The Dirac Delta Function
4. Symmetry Conditions of Fourier Transforms
5. Convolution and Correlation
6. Summary