Tutorial Solutions

9 Holography

9.1 Efficiency of Amplitude Hologram

Show that if the fringe intensity pattern of a hologram is

$$g(x,y) = g_0 + \delta g(x,y)$$

then the Amplitude transmission of a thin hologram formed from this distribution is

$$T_a(x,y) = T_0 - a\delta\hat{g}(x,y) + b(\delta\hat{g}(x,y))^2$$

where a and b are constants dependent of the film type. and

$$\delta\hat{g}(x,y) = \frac{\delta g(x,y)}{g_0}$$

When reconstructed this amplitude transmission will result in first and second order reconstructions. Calculate the ratio of power between the first and second order reconstructions, and hence estimate the required ratio of *intensities* between the *object* and *reference* beams used to form the hologram so that the first order reconstruction has 5 times the power of the second. (assume the film has $\gamma = 1.3$).

Hint: You will have to assume that $r^2 \gg o^2$ where r^2 is the reference beam intensity and o^2 the object beam intensity to get to the expression quoted in lectures.

Estimate total efficiency of such a hologram, assuming that the plate is exposed to an OD≈ 0.5. The total efficiency is defined at the intensity in the First Order reconstruction over the total input power.

Hint: Make the same assumption regarding r^2 and o^2. This part is rather difficult.

Solution

Using the notation as in lectures, we have that the intensity on the holographic plate is

$$g(x,y) = g_0 + \delta g(x,y)$$

where we have that

$$\delta g(x,y) = 2ro(x,y)\cos(\kappa x \sin\theta - \phi(x,y))$$

In coherent light the amplitude transmitted by the hologram is

$$T_a = 10^{D_0/2} E^{-\gamma/2}$$

where the exposure $E = gt$, and t is the exposure time. We can then write this as

$$T_a = K\, g^{-\gamma/2}$$

where we have collected the constants in K to give

$$K = 10^{D_0/2} t^{-\gamma/2}$$

If we then substitute for $g(x,y)$ from above, we get that

$$T_a = K(g_0 + \delta g)^{-\gamma/2} = K\, g_0^{-\gamma/2}\left[1 + \frac{\delta g}{g_0}\right]^{-\gamma/2}$$

We then substitute

$$\delta\hat{g} = \frac{\delta g}{g_0}$$

so we have that

$$T_a = K(g_0 + \delta g)^{-\gamma/2} = K\, g_0^{-\gamma/2}\left[1 + \delta\hat{g}\right]^{-\gamma/2}$$

If we have that $|\delta g| \ll g_0$, then $|\delta\hat{g}| \ll 1$, so we can expand in terms of a power series to get, to **second** order that

$$(1 - \delta\hat{g})^{-\gamma/2} = 1 - \frac{\gamma}{2}\delta\hat{g} + \frac{\gamma(\gamma+2)}{8}(\delta\hat{g})^2$$

so we get that

$$T_a = T_0 - a\delta\hat{g} + b(\delta\hat{g})^2$$

where we have that

$$\begin{aligned} T_0 &= K g_0^{-\gamma/2} \\ a &= K\frac{\gamma}{2} g_0^{-\gamma/2} \\ b &= K\frac{\gamma(\gamma+2)}{8} g_0^{-\gamma/2} \end{aligned}$$

This is a exactly as shown in lectures

On reconstruction we apply a reconstruction wave of the

$$r\exp(\imath\kappa x\sin\theta)$$

So in the *first order* reconstruction we get,

$$\begin{aligned} r\exp(\imath\kappa x\sin\theta)a\frac{\delta g}{g_0} &= a\frac{r^2}{g_0}o(x,y)\exp(\imath\phi) \\ &+ a\frac{r^2}{g_0}o(x,y)\exp\left(\imath(\kappa x\sin 2\theta - \phi)\right) \end{aligned}$$

so that the *amplitude* of the reconstruction is given by

$$\frac{ar^2 o(x,y)}{g_0}$$

In the *second order* we get, (as shown in lectures), we get terms of the form,

$$b\frac{r^3 o^2(x,y)}{g_0^2}\exp\left(\imath 2\phi(x,y)\right)\exp(-\kappa x\sin\theta)$$

so each of these terms have *amplitude* given by

$$\frac{br^3 o^2(x,y)}{g_0^2}$$

However what we measure is the *intensity* ratio, which is given by

$$R = \frac{a^2 g_0^2}{b^2 r^2 o^2(x,y)}$$

now noting that $g_0 = r^2 + o^2$, and then substituting for a and b, we (hopefully!), get that

$$R = \frac{16}{(\gamma+2)^2} \frac{\left(r^2+o^2\right)^2}{r^2 o^2}$$

Now if we assume that $r^2 \gg o^2$, then we can take the approximation that

$$\left(r^2+o^2\right)^2 \approx r^4 + 2r^2 o^2$$

then

$$R \approx \frac{16}{(\gamma+2)^2}\left[\frac{r^2}{o^2}+2\right]$$

which is the expression quoted in lectures.

If we want the first order intensity to be $\times 5$ of the second order, then $R = 5$, so if $\gamma = 1.3$, then we have that

$$r^2 \approx 5.4 o^2$$

so we require the *intensity* of the **reference beam** to be approximately $\times 5.4$ that of the **object beam**. This is approximately the ratio noted in lectures, and "just about" justifies the approximation that $r^2 \gg o^2$!

Efficiency: If the plate is exposed to an optical density of $D = 0.5$, then the *intensity* transmittance

$$T = 10^{-D} = 0.316$$

so that the *amplitude* transmittance,

$$T_a = \sqrt{T} = 0.562$$

Noting that, in the above analysis, T_0 is the *average* amplitude transmittance, then we have that

$$T_0 = T_a = 0.562$$

We have, from above, than the *intensity* of the first order is given by

$$\frac{a^2 r^4 o^2}{g_0^2}$$

while the *intensity* of the straight through beam is

$$(T_0 r)^2$$

so the ratio of the intensities of the first order to the straight through beam is

$$R = \frac{a^2 r^2 o^2}{T_0^2 g_0^2}$$

now, if we substitute for T_0 and a, then after some manipulation, we get that

$$R = \frac{\gamma^2}{4}\frac{r^2 o^2}{g_0^2}$$

i): if we assume that $r^2 \gg o^2$, then $g_0^2 \approx r^4$, so we have that

$$R \approx \frac{\gamma^2}{4}\frac{o^2}{r^2}$$

which was the expression quoted in lectures.
ii): in this case we know that $r^2 = 5.4o^2$, so we get that

$$R = \frac{\gamma^2}{4}\frac{r^2 o^2}{(r^2+o^2)^2} = 0.056$$

This says that about 5.6% of the intensity *transmitted* through the hologram is diffracted into the reconstruction. But the *intensity* transmittance of the hologram is $T = 0.316$, so the total efficiency is

$$0.056T = 0.016 = 1.6\%$$

so about 1.6% of the intensity in the reconstruction beam is diffracted into the image reconstruction.

This calculation does not include reflections from the holographic plate and scattering losses in the gelatine, which will typically result in the loss of another 30%, which gives a typical, overall efficiency of about 1%, which was the figure quoted in lectures.

9.2 Bleached Holograms

If an amplitude hologram is bleached to form a phase distribution of

$$T_a = \exp\Big(\imath\Phi(x,y)\Big)$$

where

$$\Phi(x,y) = 2\pi\frac{g(x,y)}{g_0}$$

where $g(x,y)$ and g_0 are as defined in the previous question. Show that this hologram will reconstruct in a similar manner to the amplitude hologram.

Calculate the ratio of power between the first and second order reconstructions, and hence estimate the required ratio of *intensities* between the *object* and *reference* beams used to form the hologram so that the first order reconstruction has 5 times the power of the second. Compare with with the result for the question above, and comment.

Solution

As for the amplitude case the incident intensity is

$$g(x,y) = g_0 + \delta g(x,y)$$

where we have that

$$\delta g(x,y) = 2ro(x,y)\cos(\kappa x \sin\theta - phi(x,y))$$

after bleaching the phase is given by:

$$\Phi(x,y) = 2\pi\frac{g(x,y}{g_0} = 2\pi(1+\delta\hat{g})$$

so the amplitude transmittance is given by

$$T_a = \exp(\imath\Phi(x,y)) = \exp(\imath 2\pi\delta\hat{g}(x,y))$$

If, as in the amplitude case, we assume that $|\delta g| \ll g_0$, then $|\hat{g}(x,y)| \ll 1$, so we can expand the exponential to give,

$$T_a = 1 + \imath 2\pi\delta\hat{g} - 2\pi^2(\hat{g})^2$$

This is *exactly* the same as the *amplitude* case, and again we have

$$T_a = T_0 - a\delta\hat{g} + b(\delta\hat{g})^2$$

where we have that

$$\begin{aligned} T_0 &= 1 \\ a &= -\imath 2\pi \\ b &= -2\pi^2 \end{aligned}$$

Since this in exactly the same expression as for the amplitude case we will get exactly the same set of reconstructed terms for *first* and *second* order. (See lectures for details of the working).

In the above question the expression for the ratio between the intensity of the first and second order reconstruction was calculated to be

$$R = \frac{a^2 g_0^2}{b^2 r^2 o^2(x,y)}$$

We can substitute $g_0 = r^2 + o^2$, and for the above values of a and b to get

$$R = \frac{(r^2+o^2)^2}{\pi^2 r^2 o^2}$$

so again of we assume that $r^2 \gg o^2$, we get get that

$$R \approx \frac{1}{\pi^2}\left[\frac{r^2}{o^2} + 2\right]$$

so if we want $R = 5$, then we have that

$$\frac{r^2}{o^2} \approx 47$$

so, with this system, in order that the *first* order is $\times 5$ the intensity of the *second*, the reference beam must be approximately $\times 50$ brighter than the object. This is approximately $\times 10$ more than for the *amplitude* hologram, showing that the *phase* hologram is much more non-linear, and is likely to have much larger higher order terms.

The main advantage in bleaching holograms its that the overall effiency increases since. As shown above amplitude hologram the actual hologram absorbs about 90% of the light, so a factor of in overall brightness is often obtained at the expense of higher order terms.

9.3 A Real System

Consider the system for recording a hologram shown in lectures. Assume the *intensity* efficiency of the components are:

Component	Efficiency
Mirror	90%
Microscope Objective	80%
Beam Splitter	90%
Object (onto hologram)	5%

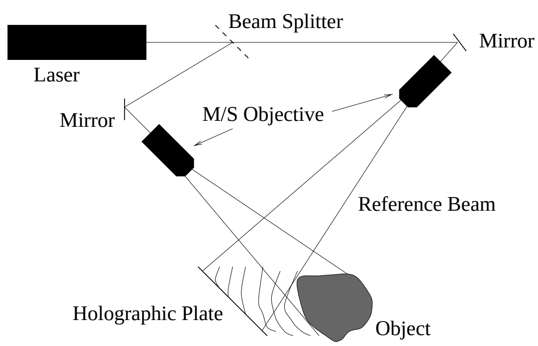
If you want the ratio of Reference Beam to Object Beam *intensity* 5:1, calculate the ratio of the beam splitter required.

If the laser has an output power of 5 mW and the holographic material requires and exposure of $20\,\mu\mathrm{J}/\mathrm{cm}^2$ *estimate* the exposures time needed to record the hologram when a) using a 35 mm film, b) using a 10×10 cm glass plate.

Hint: Ignore the fact that the laser beam is a Gaussian, and assume the recorded hologram is a circular patch the width of the film. Remember that 35 mm film has an active area of 24×36 mm.

Solution

Part 1: In the holographic system



each "arm" contains one beamsplitter, one mirror and one microscope objective, so component efficiency losses are the same for each "arm". At the hologram we require that

$$I_r \approx 5\,I_o$$

where $I_r = r^2$ and $I_o = o^2$. The object intensity, I_o, is only 5% of the object *illumination* intensity, I_i, so that:

$$I_r \approx 5 \times \frac{5}{100} I_i \qquad \frac{I_i}{I_r} \approx 4$$

so the need the object *illumination* intensity to be approximately $\times 4$ of the reference beam. So we need a beamsplitter with a 1:4 ratio, and the *stronger* beam used to illuminate the object.

Part 2: The intensity falling on the hologram (on average) is

$$I_h = I_r + I_o \approx 1.2\,I_r$$

In the reference "arm" there is one beamsplitter (1:4), one mirror and one microscope objective, so if the laser is of power I_t, then

$$I_h \approx 0.9 \times 0.25 \times 0.9 \times 0.8 \times 1.2\, I_T = 0.19\, I_t$$

a) for a 35 mm film the "hologram" is approximately 25 mm in diameter, so the area $A \approx 5\,\mathrm{cm}^2$. Exposures is

$$E = \frac{I_h \tau}{A}$$

so for $I_t = 5\,\mathrm{mW}$ and $20\,\mu\mathrm{J}/\mathrm{cm}^2$, we have that

$$\tau \approx 0.1\mathrm{secs}$$

Reasonable simple, provided object is well locked down.

b) for a $10 \times 10\,\mathrm{cm}$ plate the "hologram" is approximately 10 cm is diameter, so the area $A \approx 78\,\mathrm{cm}^2$, so the exposure is increased to

$$\tau \approx 40\mathrm{secs}$$

Sever stability problems. This is long enough for air currents and thermal changes in the object and optics to be a major problem. In practice, good holographic equipment is a good vibration free location allows exposures up to about 10 secs without major difficulties. Longer exposures need "active" stabilisation systems to compensate for thermal chances.

Experimentally, in the P4 laboratory, the typical exposure on for a 35 mm *hologram is* ≈ 0.25 secs *and for a "quarter-plate" (*5×5 cm*) is* ≈ 3 secs*. It is not possible to make a* 10×10 cm *hologram due to stability problems due to exposure times in the order of 1 minute.*

9.4 Trying to be too Clever

A "clever" student is using the holographic system from the previous question and decides to increase the object reflectivity by painting it white correction fluid (TIPP-EX). They now spend many hours totally failing to make a hologram, why?

What should they have painted it with?

Solution

This is a polarisation effect. In order for the Object and Reference beam to interfere and form the holographic fringes, the polarsiations of the two beams *must* be the same. In fact it does not matter what the state of polarisation is, but they have to be the same.

Most lasers give out linear polarised light. This polarisation is well maintained at good mirrors and lenses, so to get a good hologram the polarisation of the object illumination beam must be maintained when it is reflected off the object. Unfortunately correction fluid is a colloidal suspension of tiny particles of white pigment (like emulsion paint), which has the most unfortunate effect of randomising the polarisation on refelction. Therefore the object beam and the reference beam have different polarisation, so no interference, no holograms and a very fustrated student. *I have seen this problem really happen.*

It was a good idea to paint the object but with a metalic paint which forms a conductive surface and thus is a simple (although rough) reflector. Aluminium paint would be idea.

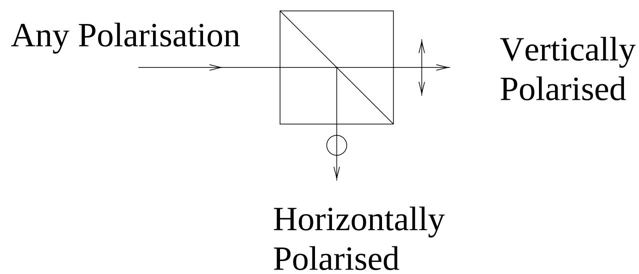
9.5 Beam splitter for Holography

In a holographic system you have to vary the beam splitter ratio depending on the object. Several optic companies make a “variable beamsplitter” which allows you to alter the beam ratios by simply rotating a knob. Suggest an explanation for this optical system.

Hint: It contains two half-wave plates and a polarising beamsplitter.

Solution

The system is based on a polarising beam splitter, which is a beam cube of quartz which splits the an input beam into *vertically* and *horizontally* polarised light as shown below:



If we take the case of the input beam being *plane* polarised at angle θ to the vertical then, from Malus’ Law the *intensity* of the vertical and horizontally transmitted beams will be,

$$I_v = I_0 \cos^2\theta \quad \text{and} \quad I_h = I_0 \sin^2\theta$$

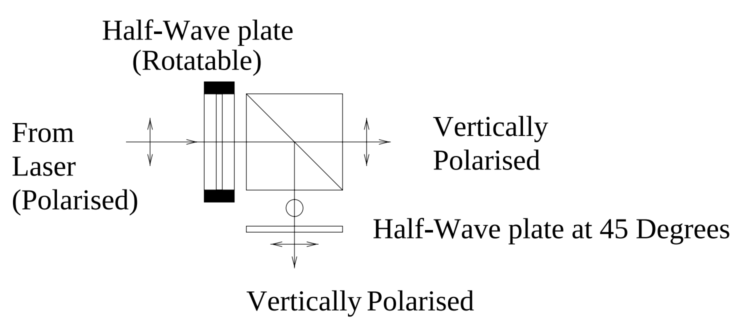
where I_0 is the input intensity. So if we can vary the angle of polarisation of the input beam the ratios of the two output beams can be varied.

The easiest way to rotate the polarisation of a plane polarised beam is with a *half-wave* plate. Remember for Physics 3 optics that a *half-wave* plate with its optical axis at angle ϕ with respect to a plane polarised beam will result in a rotation of

$$\theta = 2\phi$$

in the polarisation of the transmitted beam. So if input beam above is plane polarised it can be rotated by a rotatable half-wave plate.

For holography we need both reference and object wave to have the same polarisation, so we need to rotate one of the outputs by 90°. This is most easily done with a second half-wave plate with its optical axis at 45°. The final system is thus,



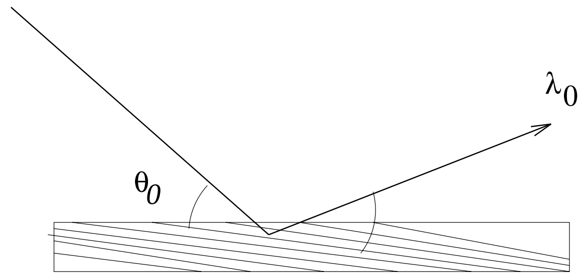
Such systems are commercially available, at very considerable cost, that give beam ratios from $\approx$ 1:10 to $\approx$ 10:1 and are ideal for holographic systems.

9.6 Multi-Image Holograms

You will notice that several of the holograms in the display on JCMB Level 4 appear to contain more than one image. Explain how this can be done.

Solution

The white light holograms on JCMB Level 4 are all *white light* holograms, so rely on the three-dimensional Bragg effects discussed is lectures. The multiple holograms (Head ⇒ Skull), and "Shakespear winking his eye" are all done by multiple exposure, and using the fact that thick holograms have a limit angle of reconstruction.

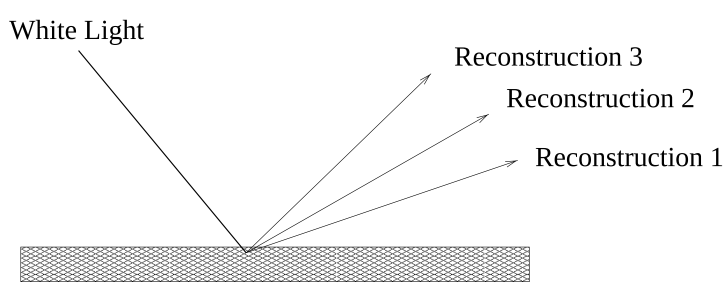


If we recond with a wavelength λ_0 then we get a reconstruction in direction θ_0 where

$$\lambda_0 = 2d \sin\theta_0$$

where d is the average separation between the planes. If we look an another angle, then we do not see any reconstrcution. If the emulsion is very thick, then the "viewing angle" (range of angles over which we can see the hologram), may be very small, frequently only a few degrees. This is usually a problem, but can be exploited to record multiple holograms on the same plate.

For example if we record *three* holograms with the object waves at different angles to the plate but the *same* reference beam, then when we reconstruct we get *three* holograms reconstructed at different angles, so as we view the hologram we see different reconstructions at different angles.



Points to note:

- The three exposures are taken at different times, so the fringe pattern associated with each in independant.
- All the holograms reconstruct at the *same* wavelength.
- There is a limit to the maximum number of holograms you can record or the holographic plate become saturated or the contarct of each set of fringes is too low to be useful.

Aside: If you want to make multi-exposure holograms that are to be hand held (for example on CD covers or credit cards), then it is better to move both the object and the reference beam on recording since the various reconstructions are viewed by tilting the hologram which changes the direction of the reconstrcution beam.

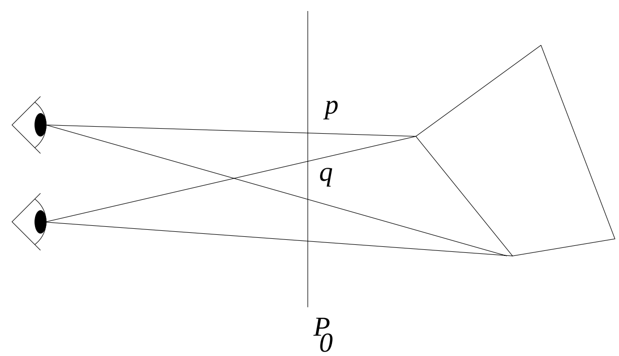
9.7 Is it Holography?

Over the last few years there has been a range of apparent 3-dimensional images, (auto-stereograms), appearing on drink cans, in newspapers, as adverts, and in poster shops. Explain how these work, and how they are related to holograms.

Hint: You have to focus your eyes "behind" the printer sheet to see the image. This, combined with a good physics training, should be enough for you to work out how they are produced and work.

Solution

The simple answer is auto-stereograms are *not* holographic since they to not rely on any diffraction phenomena during either their formation *or* reconstruction. The imaging process is basically geometric optics as follows.



Firstly the object and background are patterned with a complex repeating pattern. Then from each point on the three-D object rays from that point are traced back to the location of the observers eyes and the point where they intersect the plane P_0 recorded, for example at p and q. You then print the same intensity of dot at each each pair of points, so you end up with two overlapping view of the three-dimensional object. However since both the background and the object were patterned with the same complex repeating pattern the object structure and the "twin" effect is hidden. All this process is calculated digitally using computer graphics.

This is then viewed by "looking through" the auto-stereogram, which has the effect of setting the eyes into the same geometry are was used to calculate the pattern. The image reconstruction then relies on the human visual systems ability to "form patterns" and in particular to search for edges which are formed by matching-up the two overlapping stereo scenes. Once the eyes are in the right position the visual system then "see" two images, one in each eye, which the brain then fuses into a single three dimensional object that appears "beyond" the plate of the stereogram. (You can calculate auto-stereograms with the object "infront" by rearranging the calculation geometry).

Some points to note:

- This scheme only works if the object and background are strongly patterned so that the actual auto-stereogram has no strong lines to confuse the visual system.
- When viewing these things the eyes are in a rather un-natural position, being focused in one plane, but fixated in another plane. This is why you have to “learn” to see them, and also why your eyes become tired if you view them for any length of time.
- A good fraction of the population do not have stereo vision (they have only one working eye, or they only use one eye at a time). These people will *never* see an auto-stereogram.